The Fundamentals of Mathematical Analysis Volume 2

G.M.Fikhtengol'ts

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THE FUNDAMENTALS OF MATHEMATICAL ANALYSIS

Volume II

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CHAPTER 15

SERIES OF NUMBERS

§ 1. Introduction

234. Elementary concepts. Let there be given an infinite sequence of numbers

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots. \tag{1}$$

The symbol

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots \tag{2}$$

made up from these numbers is called an *infinite series* (or simply a *series*), and the numbers (1) themselves are called the *terms of the series.* Instead of (2), using the summation notation, we often write:

$$\sum_{n=1}^{\infty} a_n, \tag{2a}$$

the index n here taking all values from 1 to ∞†.

We will begin by adding together the terms of the series one after the other, thus forming the (infinitely many) sums:

$$A_1 = a_1,\ A_2 = a_1 + a_2,\ A_3 = a_1 + a_2 + a_3,$$
$$\ldots,\ A_n = a_1 + a_2 + \ldots + a_n,\ \ldots; \tag{3}$$

these are called the *partial sums* of the series. We shall always be comparing this sequence of partial sums $\{A_n\}$ with the series (2): the latter is itself contained in the set formed by the given sequence.

The finite or infinite limit A of the partial sum A_n of the series (2) *as* $n \to \infty$:

$$A = \lim A_n,$$

† However, it is sometimes convenient to begin a series of terms not with the index unity, but with zero or even with some natural number greater than unity.

is called the sum of the series, and we write

$$A = a_1 + a_2 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n,$$

thus attaching numerical significance to the symbol (2) or (2a). *If the series has a finite sum, it is called* convergent; *in the contrary case (i.e. if the sum is equal to* $\pm\infty$, *or there is no sum at all) it is called* divergent.

Thus the problem of whether the series (2) converges is, by definition, equivalent to the problem of whether there exists a finite limit for the sequence (3). Conversely, for any sequence

$$x_1, x_2, x_3, \ldots, x_n, \ldots,$$

the problem of whether a finite limit exists for it can be reduced to the problem of whether the series

$$x_1 + (x_2 - x_1) + (x_3 - x_2) + \ldots + (x_n - x_{n-1}) + \ldots, \qquad (4)$$

whose partial sums will be precisely the terms of this sequence, converges. Here the sum of the series coincides with the limit of the sequence.

In other words, the consideration of an infinite series and its sum is simply a new form of studying a sequence and its limit. But this form, as the reader will see from the subsequent discussion, has invaluable advantages both in establishing the very existence of a limit, and in calculating it. This fact makes infinite series a most important tool for research in mathematical analysis and its applications.

Examples. (1) We obtain the simplest example of an infinite series by summing the geometrical progression (already familiar to the reader):

$$a + aq + aq^2 + \ldots + aq^{n-1} + \ldots$$

Its partial sum (if $q \neq 1$) will be

$$s_n = \frac{a - aq^n}{1 - q}.$$

If the denominator of the progression, q, is of absolute value less than one, then [as we already know, Sec. 30, *(6)*] s_n has a finite limit

$$s = \frac{a}{1 - q},$$

i.e. our series converges and s will be its sum.

When $|q| \geqslant 1$ the same progression provides an example of a divergent series. If $q \geqslant 1$, then its sum will be $+\infty$ or $-\infty$ (according to the sign of a); in the other cases the sum does not exist at all. We note the curious series which is obtained when $a = 1$, $q = -1$:

$$1 - 1 + 1 - 1 + \ldots \equiv 1 + (-1) + 1 + (-1) + \ldots .\dagger$$

Its partial sums are alternately equal to 1 and then to 0.

(2) It is easy to establish the divergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} + \ldots .$$

In fact, because its terms decrease, its nth partial sum is

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > n \frac{1}{\sqrt{n}} = \sqrt{n}$$

and increases to infinity together with n.

(3) Finally, the variable

$$y_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$$

provides us with a less trivial example; in Sec. 49 we proved that this tends to the number e. This is equivalent to the assertion that e is the sum of the infinite series:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + \ldots = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}.$$

Recalling the approximate calculation of the number e in Sec. 49, the reader will be able to assess the advantages in this example of successively introducing progressively smaller corrections, which gradually improve the approximate values of e obtained in the form of partial sums.

235. The most elementary theorems. If in the series (2) we reject the first m terms, we obtain the series:

$$a_{m+1} + a_{m+2} + \ldots + a_{m+k} + \ldots = \sum_{n=m+1}^{\infty} a_n, \tag{5}$$

called the *remainder of the series* (2) *after the m-th term.*

---

† If any term a of the series is a negative number: $a = -b$ (where $b > 0$), then instead of $\ldots + (-b) + \ldots$ one writes $\ldots - b + \ldots$.

We emphasize that, nevertheless, $-b$ will here be a term of the series, but b will not.

(1) If the series (2) *converges, then any remainder* (5) *of it also converges; conversely, from the convergence of the remainder* (5) *follows the convergence of the original series* (2).

We fix m and denote the kth partial sum of the series (5) by A'_k:

$$A'_k = a_{m+1} + a_{m+2} + \ldots + a_{m+k}.$$

Then, clearly,

$$A'_k = A_{m+k} - A_m. \tag{6}$$

If the series (2) converges, so that $A_n \to A$, then—as k increases without limit—there also exists a finite limit

$$A' = A - A_m \tag{7}$$

for the sum A'_k, which also implies the convergence of the series (5).

Conversely, if it is given that the series (5) converges, so that $A'_k \to A'$, then we rewrite equation (6), letting $k = n - m$ (where $n > m$) in it, thus:

$$A_n = A_m + A'_{n-m};$$

hence one can see that—as n increases without limit—the partial sum A_n has the limit:

$$A = A_m + A', \tag{8}$$

i.e. the series (2) converges.

In other words, *the rejection of a finite number of the first terms of a series or the adjunction at the beginning of it of several new terms does not affect the behaviour of the series* (in the sense of its convergence or divergence).

The sum of the series (5), if it exists, we denote by the symbol α_m instead of A', indicating by this sign the term after which the remainder is taken. Then the formulae (8) and (7) can be rewritten in the following form:

$$A = A_m + \alpha_m, \quad \alpha_m = A - A_m. \tag{9}$$

If m increases to infinity, then $A_m \to A$ and $\alpha_m \to 0$. Thus:

(2) If the series (2) *converges, then the sum* α_m *of its remainder after the m-th term tends to zero as m increases.*

We mention the following simple properties of convergent series:

(3) If the terms of the convergent series (2) are all multiplied by the same factor c, then its convergence is not destroyed, and the sum is merely multiplied by c.

In fact, the partial sum $\bar{A}_n$ of the series

$$ca_1 + ca_2 + \ldots + ca_n + \ldots$$

is clearly equal to

$$\bar{A}_n = ca_1 + ca_2 + \ldots + ca_n = c(a_1 + a_2 + \ldots + a_n) = cA_n$$

and has the limit cA.

(4) Two convergent series

$$A = a_1 + a_2 + \ldots + a_n + \ldots$$

and

$$B = b_1 + b_2 + \ldots + b_n + \ldots$$

can be added or subtracted term by term, so that the series

$$(a_1 \pm b_1) + (a_2 \pm b_2) + \ldots + (a_n \pm b_n) + \ldots$$

also converge, and their sums are equal, respectively, to $A \pm B$.

If A_n, B_n, and C_n denote the partial sums of the given series, then, clearly,

$$\begin{aligned} C_n &= (a_1 \pm b_1) + (a_2 \pm b_2) + \ldots + (a_n \pm b_n) \\ &= (a_1 + a_2 + \ldots + a_n) \pm (b_1 + b_2 + \ldots + b_n) = A_n \pm B_n. \end{aligned}$$

Taking limits, we find that

$$\lim C_n = \lim A_n \pm \lim B_n,$$

which proves our assertion.

In conclusion we will make one final remark.

(5) The general term a_n of a convergent series tends to zero.

This can be proved in a quite elementary way: since A_n (and with it also A_{n-1}) has the finite limit A, then

$$a_n = A_n - A_{n-1} \to 0.$$

The preceding assertion contains *a necessary condition for the convergence of the series*, which we will often use. A series known

to violate it diverges. However, it is important to emphasize that this condition is not by itself *sufficient* for the convergence of the series. In other words, even when it is satisfied, the series may diverge. An example of this is the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},$$

considered above [Sec. 234, *(2)*]; the reader will find numerous examples of this kind in what follows.

§ 2. The convergence of positive series

236. A condition for the convergence of a positive series. Now let us consider the question of establishing the convergence or divergence of a series. This question can be answered most easily for series whose terms are non-negative; for brevity, we will simply call such a series *positive*.

Let the series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots \qquad \text{(A)}$$

be positive, i.e. $a_n \geqslant 0$ $(n = 1, 2, 3, \ldots)$. Then, clearly,

$$A_{n+1} = A_n + a_{n+1} \geqslant A_n,$$

i.e. the variable A_n is increasing. Recalling the theorem on monotonic variables [Sec. 44], we immediately arrive at the following proposition which is fundamental in the theory of positive series:

THEOREM. *The positive series* (A) *always has a sum; this sum will be finite (i.e. the series converges) if the partial sums are bounded above, and infinite (i.e. the series diverges) otherwise.*

All the practical criteria of convergence and divergence of positive series depend, basically, on this simple theorem. But only in rare cases is an immediate application of it sufficient to determine the character of the series. We give some examples of this type.

(1) We consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots,$$

known as a ***harmonic*** series†.

We have the obvious inequality:

$$\frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{2n-1} > n\frac{1}{2n} = \frac{1}{2}. \tag{1}$$

If all the terms of the harmonic series, beginning with the second, are successively set out in groups with 2, 4, 8, ... terms in each:

$$\underbrace{\frac{1}{2} + \frac{1}{3}}_{2}; \quad \underbrace{\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}}_{2^2}; \quad \underbrace{\frac{1}{8} + \ldots + \frac{1}{15}}_{2^3}; \quad \ldots;$$

$$\underbrace{\frac{1}{2^{k-1}} + \ldots + \frac{1}{2^k - 1}}_{2^{k-1}}; \quad \ldots,$$

then each of these separate sums will be greater than 1/2; it is easy to see this if one lets $n = 2, 4, 8, \ldots, 2^{k-1}, \ldots$ successively in (1). We denote the nth partial sum of the harmonic series by H_n; then, clearly,

$$H_{2^k} > k\frac{1}{2}.$$

We see that the partial sums cannot be bounded above: the series has an infinite sum.

We mention here that, as n increases, H_n increases very slowly. For instance, Euler calculated that

$$H_{1000} = 7.48 \ldots, \quad H_{1,000,000} = 14.39 \ldots, \text{ etc.}$$

Later we shall have occasion to characterize precisely the increase in the sums H_n [Sec. 238, *(4)*].

(2) Now we will consider the more general series

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots + \frac{1}{n^s} + \ldots,$$

where s is any real number; it contains the preceding series as a special case (where $s = 1$). Like the series (1), this series is also called *harmonic*.

---

† Each of its terms, beginning with the second, is the harmonic mean of the two neighbouring terms. A number c is called the harmonic mean of the numbers a and b if

$$\frac{1}{c} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right).$$

Because, when $s<1$, the terms of the series considered are larger than the corresponding terms of the series (1), then with this hypothesis the partial sums are even less bounded above, so that the series diverges.

Let us consider the case when $s>1$; for convenience we set $s=1+\sigma$, where $\sigma>0$. Analogously to (1), we now have:

$$\frac{1}{n^s}+\frac{1}{(n+1)^s}+\ldots+\frac{1}{(2n-1)^s}<n\frac{1}{n^s}=\frac{1}{n^\sigma}. \tag{2}$$

Separating, as above, successive groups of terms:

$$\underbrace{\frac{1}{2^s}+\frac{1}{3^s}}_{2};\quad \underbrace{\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\frac{1}{7^s}}_{2^2};\quad \underbrace{\frac{1}{8^s}+\ldots+\frac{1}{15^s}}_{2^3};\quad \ldots;$$

$$\underbrace{\frac{1}{(2^{k-1})^s}+\ldots+\frac{1}{(2^k-1)^s}}_{2^{k-1}};\quad \ldots;$$

and using (2) it is easy to prove that these sums are respectively less than the terms of the progression

$$\frac{1}{2^\sigma},\ \frac{1}{4^\sigma}=\frac{1}{2^{2\sigma}},\ \frac{1}{8^\sigma}=\frac{1}{2^{3\sigma}},\ \ldots,\ \frac{1}{(2^{k-1})^\sigma}=\frac{1}{2^{(k-1)\sigma}},\ \ldots.$$

In such a case it is clear that whatever partial sum of the series considered is taken, it will be less than the constant number

$$\frac{1}{1-1/2^\sigma};$$

hence, the series converges.

237. Theorems on the comparison of series. The convergence or divergence of a positive series can often be established by comparing it with another series known to be convergent or divergent. The following simple theorem is founded on the basis of such a comparison.

THEOREM 1. *Let there be given two positive series*

$$\sum_{n=1}^{\infty}a_n=a_1+a_2+\ldots+a_n+\ldots \tag{A}$$

and

$$\sum_{n=1}^{\infty}b_n=b_1+b_2+\ldots+b_n+\ldots. \tag{B}$$

If for all n sufficiently large (say for $n>N$), the inequality $a_n\leqslant b_n$ is satisfied, then from the convergence of the series (B) *follows the*

convergence of the series (A) *or—equivalently—from the divergence of the series* (A) *follows the divergence of the series* (B).

Proof. Because the rejection of a finite number of the first terms of a series does not affect its behaviour [Sec. 235, *(1)*], we can assume, without loss of generality, that $a_n \leqslant b_n$ for all the values $n = 1, 2, 3, \ldots$. Denoting the partial sums of the series (A) and (B), respectively, by A_n and B_n, we have:

$$A_n \leqslant B_n.$$

Let the series (B) converge; then, by the theorem of Sec. 236, the sums B_n are bounded:

$$B_n \leqslant L \quad (L = \text{const};\ n = 1,\ 2,\ 3,\ \ldots).$$

By virtue of the preceding inequality, it is certainly true that

$$A_n \leqslant L,$$

and this, by the same theorem, implies the convergence of the series (A).

In practice, the following theorem, which follows from the first, is sometimes more convenient:

THEOREM 2. *If there exists a limit*

$$\lim \frac{a_n}{b_n} = K^{\dagger} \qquad (0 \leqslant K \leqslant \infty),$$

then from the convergence of the series (B), *where* $K < \infty$, *follows the convergence of the series* (A), *and from the divergence of the first series, where* $K > 0$, *follows the divergence of the second.* [Thus, when $0 < K < \infty$ the series converge or diverge together.]

Proof. Let the series (B) converge and $K < \infty$. Taking an arbitrary number $\varepsilon > 0$, with the same definition of a limit, for a sufficiently large n we will have

$$\frac{a_n}{b_n} < K + \varepsilon, \quad \text{whence} \quad a_n < (K + \varepsilon) b_n.$$

By virtue of Sec. 235, *(3)*, the series $\sum (K + \varepsilon) b_n$, obtained by multiplying the terms of the series (B) by the constant number

† We presuppose here that $b_n \neq 0$.

$K+\varepsilon$, will converge together with the series (B). Hence, by the preceding theorem, the convergence of the series (A) follows.

If the series (B) diverges and $K>0$, then, in this case, the inverse ratio b_n/a_n has a finite limit; the series (A) must be divergent because, if it converged, then, by what has been proved, the series (B) would also converge.

Finally, we give one more theorem on comparison which is also a corollary of the first theorem.

THEOREM 3. *If, at least for sufficiently large n (say for $n>N$), the inequality*

$$\frac{a_{n+1}}{a_n} \leqslant \frac{b_{n+1}}{b_n}\dagger \tag{3}$$

is satisfied, then from the convergence of the series (B) *follows the convergence of the series* (A) *or, equivalently, from the divergence of the series* (A) *follows the divergence of the series* (B).

Proof. As above, in the proof of Theorem 1, without loss of generality it can be assumed that the inequality (3) is valid for *all* the values $n=1, 2, 3, \ldots$. Thus we have:

$$\frac{a_2}{a_1} \leqslant \frac{b_2}{b_1}, \quad \frac{a_3}{a_2} \leqslant \frac{b_3}{b_2}, \quad \ldots, \quad \frac{a_n}{a_{n-1}} \leqslant \frac{b_n}{b_{n-1}}.$$

Multiplying the inequalities together term by term we obtain:

$$\frac{a_n}{a_1} \leqslant \frac{b_n}{b_1} \quad \text{or} \quad a_n \leqslant \frac{a_1}{b_1} b_n \quad (n=1, 2, 3, \ldots).$$

Let the series (B) converge; the series $\sum (a_1/b_1)b_n$, obtained by multiplying its terms by the constant factor a_1/b_1, converges with it. But then, by Theorem 1, the series (A) also converges, thus proving the theorem.

We now go on to some examples of establishing the convergence or divergence of series by immediate application of the comparison theorems.

238. Examples. *(1)* $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ converges, because

† Here a_n and b_n, of course, are presupposed distinct from zero.

$$\frac{(n!)^2}{(2n)!} = \frac{n!}{2^n(2n-1)!!} < \frac{1}{2^n}$$

(Theorem 1).

(2) Comparison with the harmonic series [Sec. 236] enables one to determine the behaviour of many series. By Theorem 1:

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{[n(n^2+1)]}}$ converges: $\frac{1}{\sqrt{[n(n^2+1)]}} < \frac{1}{n^{3/2}}$;

(b) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^p}$ $(p>0)$ diverges: $(\log n)^p < n$ for sufficiently large n;

(c) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}$ converges: $\frac{1}{(\log n)^{\log n}} = \frac{1}{n^{\log\log n}} < \frac{1}{n^2}$ for sufficiently large n.

(3) By Theorem 2:

(a) $\sum_{n=1}^{\infty} \sin\frac{x}{n}$ $(0<x<\pi)$ diverges: $\sin\frac{x}{n}\Big/\frac{1}{n} \to x$; analogously, the series

$$\sum_{n=1}^{\infty} \log\left(1+\frac{x}{n}\right)(x>0), \qquad \sum_{n=1}^{\infty} (\sqrt[n]{a}-1) \quad (a>1);$$

also diverge.

(b) $\sum_{n=1}^{\infty} \left(1-\cos\frac{x}{n}\right)$ converges: $\left(1-\cos\frac{x}{n}\right)\Big/\frac{1}{n^2} \to \frac{x^2}{2}$.

(4) Finally we consider the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \log\frac{n+1}{n}\right).$$

By means of the differential calculus it is easy to establish the inequality:

$$\log(1+x) < x \qquad (x \neq 0,\ -1<x<\infty).$$

Using this we can write:

$$\log\frac{n+1}{n} = \log\left(1+\frac{1}{n}\right) < \frac{1}{n},$$

and, at the same time,

$$\log\frac{n+1}{n} = -\log\frac{n}{n+1} = -\log\left(1-\frac{1}{n+1}\right) > \frac{1}{n+1}.$$

Therefore,

$$0 < \frac{1}{n} - \log\frac{n+1}{n} < \frac{1}{n} - \frac{1}{n+1} < \frac{1}{n(n+1)} < \frac{1}{n^2}.$$

Thus the terms of the given series are positive and less than the corresponding terms of the convergent series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ [Sec. 236]; hence, the given series also converges.

If we denote its sum by C, then its partial sum is

$$\sum_{k=1}^{n}\left(\frac{1}{k} - \log\frac{k+1}{k}\right) = H_n - \log(n+1) \to C$$

(H_n denotes, as always, the partial sum of the harmonic series). Here we can replace $\log(n+1)$ by $\log n$, because their difference, equal to $\log(1+1/n)$, tends to zero. Finally, denoting by γ_n some infinitesimal quantity, we have for H_n the remarkable formula

$$H_n = \log n + C + \gamma_n. \tag{4}$$

This proves that as n increases to infinity, the partial sum H_n of the harmonic series increases as $\log n$.

The constant C which appears in formula (4) is called Euler's constant. Its numerical value (which can be computed from other considerations) is:

$$C = 0.577215\ldots.$$

239. Cauchy's and d'Alembert's tests. Comparison of a given

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots \tag{A}$$

series with standard series, known to be convergent or divergent, can also be made in another, so to speak, more restricted way.

Let us, on the one hand, take as the series (B) for comparison the convergent geometrical progression

$$\sum q^n = q + q^2 + \ldots + q^n + \ldots \qquad (0 < q < 1),$$

and, on the other hand, the divergent progression

$$\sum 1 = 1 + 1 + \ldots + 1 + \ldots.$$

Comparing the series (A), which is being tested with these series, by the method of Theorem 1, we arrive at the following test:

CAUCHY'S TEST. *For the series* (A) *we form the expression*

$$\varrho_n = \sqrt[n]{a_n}.$$

If, for sufficiently large n, *the inequality*

$$\varrho_n \leqslant q,$$

is satisfied, where q *is a constant number, less than 1, then the series converges; if, however, for sufficiently large* n

$$\varrho_n \geqslant 1$$

then the series diverges.

In fact, the inequalities $\sqrt[n]{a_n} \leqslant q$ or $\sqrt[n]{a_n} \geqslant 1$ are equivalent, respectively, to: $a_n \leqslant q^n$ or $a_n \geqslant 1$; it merely remains to apply Theorem 1†.

More often, however, this test is applicable in another, limit, form:

Let us suppose that the expression ϱ_n *has a limit (finite or not):*

$$\lim \varrho_n = \varrho.$$

Then, when $\varrho < 1$ *the series converges and when* $\varrho > 1$ *the series diverges.*

If $\varrho < 1$, then we take a positive number ε, less than $1 - \varrho$, so that $\varrho + \varepsilon < 1$. Then by the definition of a limit, for $n > N$ it will be true that:

$$\varrho - \varepsilon < \varrho_n < \varrho + \varepsilon.$$

The number $\varrho + \varepsilon$ plays the part of the number q in the preceding formula: the series converges.

If, however, $\varrho > 1$ and is finite, then, taking $\varepsilon = \varrho - 1$, so that $\varrho - \varepsilon = 1$, for sufficiently large values of n on this occasion we

† The divergence of a series, of course, can also be established simply by referring to the violation of the necessary condition for convergence [Sec. 235, *(5)*].

will have $\mathcal{C}_n > 1$, so the series diverges. There is an analogous result when $\mathcal{C} = \infty$.

In the case when $\mathcal{C} = 1$, this test does not enable us to determine the behaviour of the series.

If the comparison of the series (A) with the given standard series is performed according to Theorem 3, then we arrive at the following test:

D'ALEMBERT'S TEST. *For the series* (A) *we consider the ratio*

$$\mathcal{D}_n = \frac{a_{n+1}}{a_n}.$$

If, for sufficiently large n, *the inequality*

$$\mathcal{D}_n \leqslant q$$

is satisfied, where q *is a constant number, less than* 1, *then the series converges; if, however, for sufficiently large* n

$$\mathcal{D}_n \geqslant 1,$$

then the series diverges†.

It is more convenient, however, to make use of the limit form of the test:

Let us suppose that the ratio $\mathcal{D}_n$ *has a limit (finite or not):*

$$\lim \mathcal{D}_n = \mathcal{D}.$$

Then, when $\mathcal{D} < 1$ *the series converges, and when* $\mathcal{D} > 1$ *the series diverges.*

The proof is exactly the same as in the case of Cauchy's test.

This test also yields no result if $\mathcal{D} = 1$.

Remark. We have retained the name d'Alembert in explaining this test because this is usually done. But, in fact, d'Alembert did not give an exact presentation of the convergence of a series and of its sum as the limit of partial sums. D'Alembert warns us about making use of series in which the ratio of a term to the term preceding it eventually becomes absolutely greater than one, and considers such series "faulty". In order that a series should be "well-behaved and not faulty", he requires only that the given ratio should finally become (absolutely) less than one; we emphasize that we required something else namely that this ratio

† Here also divergence follows directly from the violation of the necessary condition for convergence: in fact, if $a_{n+1}/a_n \geqslant 1$ or $a_{n+1} \geqslant a_n$, then a_n cannot tend to zero.

should become less than a constant proper fraction q! D'Alembert's condition is not sufficient for the convergence of a series in the contemporary sense: for instance, it is satisfied for the harmonic series $\sum_{n=1}^{\infty} 1/n$, which is known to be divergent.

In the limit form the test was first accurately formulated and proved by Cauchy.

Examples. (1) Let there be given a series $\sum_{n=1}^{\infty} (1-1/n)^{n^2}$. It is natural to apply Cauchy's test to it:

$$\mathcal{C}_n = \left(1-\frac{1}{n}\right)^n, \quad \mathcal{C} = \frac{1}{e} < 1,$$

so that the series converges.

(2) We apply d'Alembert's test to the following series:

(a) $$1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} (x>0); \quad \mathcal{D}_n = \frac{x}{n+1}, \quad \mathcal{D} = 0,$$

the series converges, whatever the value of x;

(b) $$\sum_{n=1}^{\infty} n! \left(\frac{x}{n}\right)^n (x>0); \quad \mathcal{D}_n = \frac{x}{(1+1/n)^n}, \quad \mathcal{D} = \frac{x}{e},$$

the series converges when $x<e$ and diverges when $x>e$; when $x = e$, d'Alembert's test in the limit form yields no result, but because $(1+1/n)^n < e$ so $\mathcal{D}_n > 1$ always, the series diverges.

(3) The given tests are not applicable to the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \ldots + \frac{1}{n^s} \ldots \quad (s>0). \qquad (H_s)$$

In fact, in this case $\mathcal{C}_n = 1/n^{s/n} < 1$, but $\lim \mathcal{C}_n = 1$, so that

$$\log \mathcal{C}_n = -s\frac{\log n}{n} \to 0$$

[Sec. 121, *(3)*]. Analogously, $\mathcal{D}_n = 1/(1+1/n)^s < 1$, but, clearly, $\lim \mathcal{D}_n = 1$ also.

From other considerations [Sec. 236] we know, however, that the harmonic series converges when $s>1$ and diverges when $s \leqslant 1$.

240. Raabe's test. In those cases when the simple tests given do not provide the answer, it is appropriate to have recourse to more complex tests, based on the comparison of the series being examined with another standard series converging, so to speak, "more slowly" or diverging "more slowly" than a progression.

We consider here Raabe's test†, in which the given series (A) is, in fact, compared with the harmonic series (H_s). For this it is useful to consider the following expression:

$$\mathcal{R}_n = n\left(1 - \frac{a_{n+1}}{a_n}\right).$$

RAABE'S TEST. *If, for sufficiently large n, the inequality*

$$\mathcal{R}_n \geqslant r$$

is satisfied, where r is a constant greater than 1, then the series converges; if, however, for sufficiently large n,

$$\mathcal{R}_n \leqslant 1,$$

then the series diverges.

Thus, for sufficiently large n let us have

$$n\left(1 - \frac{a_{n+1}}{a_n}\right) \geqslant r > 1 \quad \text{or} \quad \frac{a_{n+1}}{a_n} \leqslant 1 - \frac{r}{n}.$$

Now we introduce some number s between 1 and r: $1 < s < r$. Because—by the known limit relation [Sec. 65, *(3)*]—

$$\lim_{n\to\infty} \frac{(1-1/n)^s - 1}{-1/n} = s,$$

then for sufficiently large n it will be true that [Sec. 36, *(1)*]

$$\frac{(1-1/n)^s - 1}{-1/n} < r \quad \text{or} \quad \left(1 - \frac{1}{n}\right)^s > 1 - \frac{r}{n},$$

and hence also

$$\frac{a_{n+1}}{a_n} < \left(1 - \frac{1}{n}\right)^s.$$

This inequality can be rewritten in the following form

$$\frac{a_{n+1}}{a_n} < \left(\frac{n-1}{n}\right)^s = \frac{1/n^s}{1/(n-1)^s}.$$

On the right we have the ratio of two positive terms of a convergent series (H_s) ($s > 1$!); applying Theorem 3, we can see that series (A) converges.

† Joseph Ludwig Raabe (1801–1859) was a German mathematician.

If, however, for sufficiently large n,

$$n\left(1-\frac{a_{n+1}}{a_n}\right)\leqslant 1,$$

then from this we find at once that

$$\frac{a_{n+1}}{a_n}\geqslant\frac{n-1}{n}=\frac{1/n}{1/(n-1)};$$

applying Theorem 3 to the series (A) and to the divergent series (H_1), we deduce the divergence of the series (A).

Raabe's test is also applicable for the most part in the limit form. *Let us suppose that the expression $\mathcal{R}_n$ has a limit (finite or not):*

$$\lim \mathcal{R}_n=\mathcal{R}.$$

Then when $\mathcal{R}>1$ the series converges, and when $\mathcal{R}<1$ the series diverges.

Comparing d'Alembert's and Raabe's tests, we see that the latter is considerably stronger than the former. If the limit $\mathcal{D}=\lim \mathcal{D}_n$ exists and is distinct from 1, then for $\mathcal{R}_n=n(1-\mathcal{D}_n)$ there exists a limit $\mathcal{R}$, equal to $+\infty$ when $\mathcal{D}<1$ and to $-\infty$ when $\mathcal{D}>1$. Thus, although d'Alembert's test gives an answer to the question of how the given series behaves, Raabe's test also states that, in addition to this, all such cases are covered by two possible values of $\mathcal{R}$, namely $\pm\infty$. All remaining values of $\mathcal{R}$ (excluding $\mathcal{R}=1$), which also give an answer to the question of convergence, thus correspond to those cases when d'Alembert's test is known not to provide the answer, because $\mathcal{D}=1$.

As an example of the application of Raabe's test we consider the series:

$$1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{1}{2n+1}.$$

D'Alembert's test is not applicable to this series, because $\mathcal{D}_n=\dfrac{(2n-1)^2}{2n(2n+1)}\to 1$ (and moreover $\mathcal{D}_n<1$). We form Raabe's expression:

$$\mathcal{R}_n=n\left(1-\frac{(2n-1)^2}{2n(2n+1)}\right)=\frac{6n-1}{2(2n+1)}.$$

Because $\mathcal{R}=\lim \mathcal{R}_n=3/2>1$, the series converges.

However, when $\mathcal{R} = 1$, once again we do not have the answer to the question of the behaviour of the series. In such cases (which are very rare) it is appropriate to have recourse to even more subtle and complex tests.

241. The Maclaurin–Cauchy integral test. In conclusion we introduce one final test, differing in form from the preceding ones.

Let the series considered have the form

$$\sum_{n=1}^{\infty} a_n \equiv \sum_{n=1}^{\infty} f(n), \tag{5}$$

where $f(n)$ is the value at $x = n$ of some function $f(x)$, defined for $x \geqslant 1$†; we presuppose this function to be continuous, positive and monotonic decreasing.

We consider some primitive function for $f(x)$:

$$F(x) = \int f(x)\,dx.$$

Because its derivative $F'(x) = f(x) > 0$, then $F(x)$ increases with x and as $x \to \infty$ certainly has a limit L, finite or not [Sec. 47]. In order to distinguish these cases there has also been constructed the following test, which was expressed in geometrical form as long ago as 1742 by Maclaurin, but was left unnoticed, and was rediscovered in 1827 by Cauchy.

THE INTEGRAL TEST. *The series* (5) *converges if the limit* L *of the primitive function* $F(x)$ *is finite, and diverges if* $L = \infty$.

Clearly, it makes no difference which of the primitives is being discussed, because any two primitives must differ only by a constant; we set

$$F(x) = \int_1^x f(x)dx. \tag{6}$$

From the monotonicity of the function $f(x)$, for $n \leqslant x \leqslant n+1$,

$$a_{n+1} = f(n+1) \leqslant f(x) \leqslant f(n) = a_n,$$

† The first value of the number n, instead of 1, can also be any other natural number n_0; then the function $f(x)$ must also be considered for $x \geqslant n_0$.

so, that also,

$$a_{n+1} \leqslant \int_n^{n+1} f(x)dx \leqslant a_n.$$

For the series

$$\sum_{n=1}^{\infty} \int_n^{n+1} f(x)dx$$

the nth partial sum is

$$\int_1^{n+1} f(x)dx = F(n+1);$$

it is clear that this series converges or diverges exactly when L is finite or not. For finite L, by the "Comparison Theorem" 1 [Sec. 237], the series with smaller terms, $\sum_1^{\infty} a_{n+1}$, converges, and with it the series (5); on the other hand, when $L = \infty$ the series with the larger terms, $\sum_1^{\infty} a_n$, i.e. the series (5), certainly also diverges.

Examples. (1) We consider first of all the harmonic series:

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{1}{n}, \qquad \text{(b)} \quad \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (s > 1).$$

In the case (a):

$$f(x) = \frac{1}{x}, \quad F(x) = \int \frac{dx}{x} = \log x \to \infty \quad \text{as} \quad x \to \infty,$$

so the series diverges.

In the case (b):

$$f(x) = \frac{1}{x^s}, \quad F(x) = \int \frac{dx}{x^s} = -\frac{1}{s-1} x^{-(s-1)} \to 0 \quad \text{as} \quad x \to \infty,$$

and the series converges.

To these results (which we already know) we add a new one:

(2) $$\sum_{n=2}^{\infty} \frac{1}{n \log n}.$$

Here

$$f(x) = \frac{1}{x \log x}, \quad F(x) = \int \frac{dx}{x \log x} = \log \log x \to \infty \quad \text{as} \quad x \to \infty,$$

so the series diverges.

We note that the limit of the integral (6) as $x \to \infty$ is called the integral from 1 to ∞ † and is denoted by

$$F(\infty) = \int_1^{\infty} f(x)dx.$$

Thus, one can say that *the series* (5) *converges or diverges according to whether we have a finite value for this integral or not.*

The integral test in such a form permits a simple geometric interpretation, related to Maclaurin's idea. If one represents the function $f(x)$ as a curve (Fig. 1), then the integral $F(x)$ will express

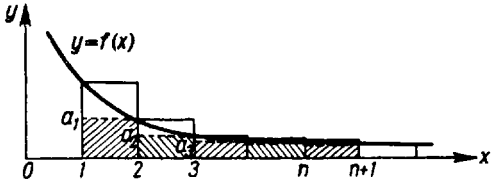

FIG. 1.

the area of the figure bounded by this curve, the axis of x and two ordinates; the integral $F(\infty)$, in a sense, can be regarded as the expression for the area of the whole figure under the curve, extended infinitely on the right. On the other hand, the terms $a_1, a_2, \ldots, a_n, \ldots$ of the series (5) express the magnitudes of the ordinates at the points $x = 1, 2, \ldots, n, \ldots$ or, equivalently, the areas of the rectangles with base 1 and with heights equal to the given ordinates.

Thus, the sum of the series (5) is none other than the sum of the areas of the externally drawn rectangles, and differs only by the first term from the sum of the areas of the enclosed rectangles. This makes completely intuitive the result established above: if the area of the curvilinear figure is finite, then the area of the stepped figure enclosed by it is obviously finite, and the series considered converges; if, however, the area of the curvilinear figure is infinite, then the area of the stepped figure containing it is also infinite, so that in this case the series diverges.

† This is the so-called improper integral; we will discuss such integrals in Chapter 17.

§ 3. The convergence of arbitrary series

242. The principle of convergence. We turn to the question of the convergence of series whose terms can have arbitrary sign. We know [Sec. 234] that—with the same definition—the question of the convergence of the series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + a_{n+1} + \ldots a_{n+m} + \ldots$$

is identical with the question of the existence of a finite limit for the sequence of partial sums

$$A_1,\ A_2, \ldots, A_n,\ A_{n+1}, \ldots, A_{n+m}, \ldots. \tag{1}$$

In Sec. 52 we established the *principle of convergence* for a sequence by giving a general condition necessary and sufficient for the existence of such a limit. Rewriting this principle of convergence in a form suitable for the sequence (1), one can now formulate it thus:

In order that the series (A) *should converge it is necessary and sufficient that to every number* $\varepsilon > 0$ *there should correspond a number* N *such that when* $n > N$ *the inequality*

$$|A_{n+m} - A_n| = |a_{n+1} + a_{n+2} + \ldots + a_{n+m}| < \varepsilon \tag{2}$$

is satisfied whatever the value of $m = 1, 2, 3, \ldots$†.

(Here $n + m$ plays the part of the second number n' in Sec. 52; it does not depend on n and, without loss of generality, can be taken to be larger than n.)

If, presupposing that the series converges, we set in particular $m = 1$ in the inequality (2), then we obtain:

$$|a_{n+1}| < \varepsilon \qquad (\text{when } n > N),$$

so that $a_{n+1} \to 0$ or (what is the same thing) $a_n \to 0$, and we are brought back to the necessary condition in Sec. 235, *(5)*. This, by itself, does not ensure the convergence of the series; indeed, it falls far short of fulfilling the requirements of the condition stated above: it is necessary not only that the sufficiently remote terms of the series, separately, be as small as desired, but also that the sums of

† The originators of this condition—Bolzano and Cauchy—both gave it as suitable for application in the problem of the convergence of an infinite series.

such terms be as small as desired, independently of how many of them are taken. This is well illustrated by the example of the harmonic series $\sum_{n=1}^{\infty} 1/n$, whose general term ($= 1/n$) tends to zero, but at the same time, if one takes the following $n-1$ terms after this term, then their sum is $> 1/2$ [Sec. 236, *(1)*].

243. Absolute convergence. Since an immediate application of the principle of convergence usually involves difficulties, it is of interest to study the type of case when the problem is solved by more simple means.

We have seen in the preceding section that in the case of positive series, convergence, for the most part, can easily be established, because of the existence of a series of convenient tests. Therefore, it is natural to begin with those cases where the question of the convergence of a given series can be reduced to the question of the convergence of a positive series.

If the terms of a series are not all positive, but from some point onwards, become positive, then by rejecting a sufficient number of the first terms of the series [Sec. 235, *(1)*] we reduce the problem to the investigation of a positive series. If the terms of the series are negative, or at least become negative from some point onwards, then we can reduce it to one of the forms already considered by changing the signs of all the terms [Sec. 235, *(3)*]. Thus, the new case will be that when amongst the terms of the series there is an infinite number both of positive and of negative terms.

We give the following definition: *if the series*

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots \qquad \text{(A)}$$

converges simultaneously with the series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + \ldots + |a_n| + \ldots, \qquad \text{(A*)}$$

then the series (A) *is said to be absolutely convergent.*

The following theorem holds.

CAUCHY'S THEOREM. *The convergence of the series* (A*) *alone, composed of the absolute magnitudes of the terms of the given series* (A), *implies the convergence of this latter series.*

The proof is obtained at once from the principle of convergence: the inequality

$$|a_{n+1}+a_{n+2}+\ldots+a_{n+m}|\leqslant|a_{n+1}|+|a_{n+2}|+\ldots+|a_{n+m}|$$

shows that if the condition for convergence is satisfied for the series (A*), then it will certainly be satisfied for the series (A).

One can also reason differently. We consider separately the positive terms of the series (A) and the absolute magnitudes of its negative terms. Numbering each of these in the order in which the terms occur in the series (A), we make up the two (positive) series

$$\sum_{k=1}^{\infty} b_k = b_1+b_2+\ldots \quad \text{(B)}, \qquad \sum_{m=1}^{\infty} c_m = c_1+c_2+\ldots \quad \text{(C)}.$$

From the convergence of the series (A*) follows the convergence of both these series. In fact, any partial sum B_k or C_m of one or other of them is smaller than some partial sum of the series (A*) and hence is less than the sum A^* of this series [Sec. 236].

If we now take the partial sum A_n of the series (A), it can be represented in the form of the difference

$$A_n = B_k - C_m,$$

where k and m denote, respectively, the number of positive or negative summands in the composition of the sum A_n, so that k and m depend on n and tend to infinity with n†. Then, clearly, there exists a finite limit

$$A = \lim A_n = B - C, \tag{3}$$

where B and C are the sums of the series (B) and (C), and the convergence of the series (A) is proved. At the same time, in the process of doing this we have also established the useful assertion that, with the assumptions we have made, the sum of the given

† All the time we have in view the case when amongst the terms of the series (A) there may be infinitely many positive and negative terms.

series is equal to the difference between the sum of the series composed of its individual positive terms and the sum of the series composed of the absolute magnitudes of its negative terms.

Thus the convergence of the series (A*) is by itself sufficient for the absolute convergence of the series (A).

As we will see below, it is possible for there to be cases when the series (A) converges and the series (A*) does not. *Then the series* (A) *is said to be non-absolutely convergent.*

In order to establish the absolute convergence of the series (A) one can apply to the positive series (A*) any of the tests for convergence studied in the previous section. However, it is necessary to be careful when using the tests for divergence: even if the series (A*) turns out to be divergent, the series (A) may still converge (non-absolutely). Only Cauchy's and d'Alembert's tests are exceptions in that, when they establish the divergence of the series (A*), this means that the general term $|a_n|$ of the series (A*) does not tend to zero†, and thus neither does a_n tend to zero, so that the series (A) also diverges. Therefore the tests just mentioned can be rewritten in a form suitable for arbitrary series. We will do this, by way of example, for d'Alembert's test:

D'ALEMBERT'S TEST. *Let there exist for the ratio* $\mathcal{D}_n^* = |a_{n+1}|/|a_n|$ *a definite limit:*

$$\mathcal{D}^* = \lim \mathcal{D}_n^* ;$$

then, when $\mathcal{D}^* < 1$ *the given series* (A) *converges absolutely, and when* $\mathcal{D}^* > 1$ *it diverges.*

Examples. (1) We apply d'Alembert's test to the series (a) and (b) which were discussed in Sec. 239, *(2)*, but we reject the requirement $x > 0$. We obtain that

(a) the series converges absolutely for all values of x;

(b) the series converges absolutely for $-e < x < e$, and diverges for $x \geqslant e$ or $x \leqslant -e$ (when $x = \pm e$ the necessary condition for convergence is violated).

(2) For the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \pm \frac{x^n}{n} \mp \ldots$$

† Cf. footnote on p. 14.

we have:

$$\mathcal{D}_n^* = \frac{n}{n+1}|x|, \quad \mathcal{D}^* = |x|,$$

so that the series converges absolutely for $|x| < 1$, and diverges for $|x| > 1$. When $x = -1$ the series also diverges, because it is obtained from the harmonic series by changing the signs of its terms. When $x = 1$ we arrive at the series

$$1 - \frac{1}{2} + \frac{1}{3} - \ldots + (-1)^{n-1}\frac{1}{n} + \ldots,$$

with regard to which the question of convergence, for the moment, remains open.

244. Alternating series. With the help of the tests considered in the preceding section, if it is possible to establish any sort of convergence, it is only the absolute convergence of a series—these tests are known not to be applicable to non-absolutely convergent series. Therefore we will consider, in conclusion, the question of the convergence of one special, but important, type of series, the so-called alternating series, amongst which there are found cases of non-absolute convergence.

An alternating series is a series whose terms have in turn first a positive and then a negative sign. An alternating series is more conveniently written in such a way that the signs of its terms are shown, for instance (if we take all $c_n > 0$),

$$c_1 - c_2 + c_3 - c_4 + \ldots + (-1)^{n-1}c_n + \ldots. \tag{4}$$

For a similar series Leibniz, as early as 1714, stated the following simple theorem (in a letter to Johann Bernoulli):

LEIBNIZ'S THEOREM. *If the terms of the alternating series* (4) *are monotonic decreasing in their absolute values:*

$$c_{n+1} < c_n \qquad (n = 1, 2, 3, \ldots) \tag{5}$$

and tend to zero:

$$\lim c_n = 0,$$

then the series converges.

Proof. The partial sums of even order C_{2m} can be written in the form:

$$C_{2m} = (c_1 - c_2) + (c_3 - c_4) + \ldots + (c_{2m-1} - c_{2m}).$$

Because every bracket, by virtue of (5), is a positive number, then from this it is clear that as m increases the sum C_{2m} also increases. On the other hand, if we write C_{2m} thus:

$$C_{2m} = c_1 - (c_2 - c_3) - \ldots - (c_{2m-2} - c_{2m-1}) - c_{2m},$$

then it is easy to see that C_{2m} remains bounded above:

$$C_{2m} < c_1.$$

In such a case, by the theorem on monotonic variables [Sec. 44], as m increases without limit the partial sum C_{2m} has a finite limit

$$\lim_{m \to \infty} C_{2m} = C.$$

Now considering the partial sums of odd order C_{2m-1}, we have, clearly, $C_{2m-1} = C_{2m} + c_{2m}$. Because the general term tends to zero, then also

$$\lim_{m \to \infty} C_{2m-1} = C.$$

Hence it follows that C will also be the sum of the given series.

Remark. We saw that the partial sums of even order C_{2m} tend to the sum C as they increase. Writing C_{2m-1} in the form

$$C_{2m-1} = c_1 - (c_2 - c_3) - \ldots - (c_{2m-2} - c_{2m-1}),$$

it is easy to establish that the sums of odd order tend to C as they decrease. Thus it is always true that

$$C_{2m} < C < C_{2m-1}.$$

In particular, we can assert that

$$0 < C < c_1. \tag{6}$$

This enables us to give a very simple and convenient value for the remainder of the series considered (which is itself an alternating series): namely,

$$\gamma_n = (-1)^{n-1}\{c_n - c_{n+1} + \ldots\}.$$

By virtue of (6) the expression in brackets is positive and less than c_n.

Thus, in all cases, *the remainder of a series of the Leibniz type*† *has the sign of its first term and is less than it in absolute magnitude.*

† This is what we shall call an alternating series which satisfies the conditions of Leibniz's theorem.

This remark is used in part in approximate calculations by means of a series [see Sec. 260].

The simplest examples of series of the Leibniz type are the series

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \ldots + (-1)^{n-1}\frac{1}{n} + \ldots$$

and

$$\text{(b)} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots + (-1)^{n-1}\frac{1}{2n-1} + \ldots.$$

The convergence of both series follows from the above theorem.

At the same time, the series formed from the absolute values of their terms will diverge: for the series (a) this will be the harmonic series, and for (b) we obtain the series

$$1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} + \ldots,$$

the divergence of which is clear from the fact that its partial sums are

$$\sum_{k=1}^{n} \frac{1}{2k-1} > \sum_{k=1}^{n} \frac{1}{2k} = \frac{1}{2} H_n.$$

Thus the series (a) and (b) provide our first examples of non-absolutely convergent series. [Below we will establish that the sum of the first of these is log 2, and the sum of the second is equal to $\pi/4$ (see Sec. 256, (22) and Sec. 255, (20)).]

§ 4. The properties of convergent series

245. The associative property. The concept of the sum of an infinite series differs essentially from the concept of the sum of a finite number of summands (considered in arithmetic or algebra) in that it contains within itself the concept of the transition to a limit. Although some properties of ordinary sums carry over to the sums of infinite series, usually they only hold when certain definite conditions are satisfied, which must also be studied. In our examples the usual properties of sums are violated in a striking way, so that, in general, it is necessary to exercise great care.

We consider the convergent series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots; \tag{A}$$

the sequence of its partial sums

$$A_1, A_2, \ldots, A_n, \ldots \tag{1}$$

converges to the sum of the series A. We begin by collecting the terms of the series into arbitrary groups, not changing their order in so doing:

$$a_1 + \ldots + a_{n_1}, \quad a_{n_1+1} + \ldots a_{n_2}, \ldots, \quad a_{n_{k-1}+1} + \ldots + a_{n_k}, \ldots.$$

Here $\{n_k\}$ is some partial, increasing sequence of numbers, extracted from the series of natural numbers.

THEOREM. *The series composed of these sums, viz.*

$$(a_1 + \ldots + a_{n_1}) + (a_{n_1+1} + \ldots + a_{n_2}) + \ldots + (a_{n_{k-1}+1} + \ldots + a_{n_k}) + \ldots, \tag{$\tilde{A}$}$$

always converges and has the same sum A as the original series. In other words, *a convergent series possesses the associative property.*

In fact, the sequence of partial sums of the new series

$$\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_k, \ldots$$

is none other than the partial sequence

$$A_{n_1}, A_{n_2}, \ldots, A_{n_k}, \ldots$$

for the sequence (1) and hence converges to the same limit A.

We notice, by the way, a complete analogy with ordinary sums; but this analogy breaks down if we try to apply the associative property, so to speak, in the reverse direction. If a convergent series ($\tilde{A}$) is given, whose terms separately represent the sum of a finite number of summands, then, if we omit the brackets, we obtain a new series (A) which may turn out to be divergent. A simple example: the series

$$(1-1) + (1-1) + (1-1) + \ldots$$

clearly converges, whilst the series

$$1 - 1 + 1 - 1 + 1 - 1 + \ldots$$

(obtained from it by omitting the brackets) will diverge.

Finally, *if—omitting the brackets—we obtain a convergent series* (A), *then its sum is the same as that of the series* ($\tilde{A}$). This follows from what has been proved above.

Remark. Under certain conditions it is possible to guarantee in advance that the series (A) will converge. The simplest case of this kind is when all the summands in ($\tilde{A}$) inside each particular set of brackets have the same sign†.

In fact, in that case, as n changes from n_{k-1} to n_k the partial sum A_n will change monotonically, hence it will be contained between $A_{n_{k-1}} = \tilde{A}_{k-1}$ and $A_{n_k} = \tilde{A}_k$. For sufficiently large k these last sums differ by an arbitrarily small amount from the sum $\tilde{A}$ of the series ($\tilde{A}$), hence the same is true also with respect to the sum A_n—for sufficiently large n—so that $A_n \to A$.

246. The permuting property of absolutely convergent series. Let there be given a convergent series (A) having the sum A. If we permute its terms in an arbitrary way, we obtain a new series:

$$\sum_{k=1}^{\infty} a'_k = a'_1 + a'_2 + \ldots + a'_k + \ldots . \qquad (A')$$

Each term a'_k of this series is identical with some definite term a_{n_k} of the original series‡.

The question arises whether the series (A′) converges and—if it converges—whether its sum will be equal to the sum A of the original series. In considering this question it will be necessary for us to draw a sharp distinction between absolutely and non-absolutely convergent series.

DIRICHLET'S THEOREM§. *If a series* (A) *converges absolutely, then the series* (A′), *obtained from it by a permutation of its terms, also converges and has the same sum A as the original series.* In other words, *an absolutely convergent series possesses the permuting property.*

Proof. We will carry out the proof in two steps.

(a) We presuppose in the first place that the series (A) is positive.

† This sign may change from one set of brackets to another.

‡ Moreover, the sequence of numbers $\{n_k\}$ reproduces—apart from order—the series of natural numbers without omissions or repetitions.

§ Peter Gustave Lejeune Dirichlet (1805–1859) was a German mathematician.

We consider an arbitrary partial sum A'_k of the series (A'). Because

$$a'_1 = a_{n_1},\ a'_2 = a_{n_2},\ \ldots,\ a'_k = a_{n_k},$$

then, taking n' greater than all the numbers $n_1, n_2, \ldots, n_k$, clearly we will have $A'_k \leqslant A_{n'}$, and hence

$$A'_k \leqslant A.$$

In this case (A') will be convergent [Sec. 236], and its sum A' will not exceed A:

$$A' \leqslant A.$$

But the series (A) is also obtained from (A') by a permutation of its terms, therefore analogously:

$$A \leqslant A'.$$

Comparing the relations obtained, we arrive at the required equation:

$$A' = A.$$

(b) Now let (A) be an arbitrary absolutely convergent series. Because the convergent positive series:

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + \ldots + |a_n| + \ldots, \qquad (A^*)$$

by what has been proved above, remains convergent under any permutation of its terms, then by the theorem in Sec. 243 the series (A) also keeps its (absolute) convergence.

Further, we saw in Sec. 243 that, in the case of the absolute convergence of the series (A), its sum can be expressed thus:

$$A = B - C,$$

where B and C are the sums of the positive series (B) and (C), composed, respectively, of the positive terms of the series (A) and of the absolute magnitudes of its negative terms. A permutation of the terms in the series (A) also causes a permutation of the terms in these series, but (by what has been proved) does not affect the sums B and C. Hence the sum of the series (A) will also remain unchanged, so proving the theorem.

247. The case of non-absolutely convergent series. Now we turn to the study of non-absolutely convergent series, and we establish that *they do not possess the permuting property.*

We make one preliminary remark. We presuppose that the series (A) converges, but not absolutely. From the convergence it follows that $\lim a_n = 0$ [Sec. 235, *(5)*]. As far as the series

$$\sum_{k=1}^{\infty} b_k \qquad \text{(B)}, \qquad\qquad \sum_{m=1}^{\infty} c_m \qquad \text{(C)},$$

which we mentioned in the preceding section, are concerned, although clearly

$$\lim_{k\to\infty} b_k = 0 \quad \text{and} \quad \lim_{m\to\infty} c_m = 0, \tag{2}$$

yet in the given case they both diverge. In fact, if among the first n terms of the series (A) there are k positive and m negative terms, then

$$A_n = B_k - C_m\,, \qquad A_n^* = B_k + C_m.$$

The second of these equations shows that both the series (B) and (C) cannot converge at the same time, because otherwise the series (A*) would also converge, contrary to our hypothesis.

Now we prove the following interesting theorem:

RIEMANN'S THEOREM. *If the series* (A) *converges non-absolutely, then no matter what number L we take in advance, finite or equal to $\pm\infty$, we can so permute the terms in this series that the transformed series has as its sum precisely the number L.*

Proof. First we will consider the case of finite L. We begin with the remark that, from Sec. 235, *(1)*, all remainders of the series (B) and (C) will also be divergent, so that in each of these series, beginning at any point, we can take enough terms that their sum would exceed any number.

Making use of this, we perform a permutation of the terms of the series (A) in the following way.

In the first place we will take so many positive terms of our series, in the order in which they occur in the series, that their sum exceeds the number L:

$$b_1 + b_2 + \ldots + b_{k_1} > L.$$

Immediately after them we write out negative terms (in the order in which they occur in the given series), taking enough of them that the common sum is less than L:

$$b_1 + b_2 + \ldots + b_{k_1} - c_1 - c_2 - \ldots - c_{m_1} < L.$$

After this we again write positive terms chosen from those which remain, so that

$$b_1 + \ldots + b_{k_1} - c_1 - \ldots - c_{m_1} + b_{k_1+1} + \ldots + b_{k_2} > L.$$

Then we take enough negative terms from those which remain, so that

$$b_1 + \ldots + b_{k_1} - c_1 - \ldots - c_{m_1} + b_{k_1+1} + \ldots + b_{k_2} - c_{m_1+1} - \ldots - c_{m_2} < L,$$

etc. We think of this process as being continued to infinity; clearly every term of the series (A), together with its sign, occurs in a definite place.

If every time we write down the terms b or c we take them no larger than is necessary to obtain the required inequalities, then the deviation from the number L on one side or the other will not exceed the absolute value of the last term written down. Then from (2) it is clear that the series

$$(b_1 + \ldots + b_{k_1}) - (c_1 + \ldots + c_{m_1}) + \ldots + (b_{k_{i-1}+1} + \ldots + b_{k_i}) - (c_{m_{i-1}+1} + \ldots + c_{m_i}) + \ldots$$

has as its sum the number L. By virtue of the remark in Sec. 245, this also remains true after removing the brackets.

If $L = +\infty$, then the set of positive terms may be made subject to the requirement that the sums become successively greater than 1, 2, 3, etc., and only one negative term at a time be placed after each group of positive terms. In this way, clearly, a series would be built up whose sum is $+\infty$. Analogously, one can also obtain a series with the sum $-\infty$.

The result obtained emphasizes the fact that non-absolute convergence is brought about only because of the mutual cancelling out of the positive and negative terms, and in particular, therefore, it depends essentially on the order in which they follow one another, whereas absolute convergence is based on the rapidity with which these terms decrease—and does not depend on their order.

Example. We consider the series, known to be non-absolutely convergent:

$$\underbrace{1 - \frac{1}{2}} + \underbrace{\frac{1}{3} - \frac{1}{4}} + \ldots + \underbrace{\frac{1}{2k-1} - \frac{1}{2k}} + \ldots, \tag{3}$$

whose sum, as we saw above, is log 2†. Rearranging its terms so that after one positive term there are two negative ones:

$$1-\underbrace{\frac{1}{2}-\frac{1}{4}}+\underbrace{\frac{1}{3}-\frac{1}{6}-\frac{1}{8}}+\ldots+\underbrace{\frac{1}{2k-1}-\frac{1}{4k-2}-\frac{1}{4k}}+\ldots, \tag{4}$$

we confirm that the sum of the series when the terms have been thus rearranged is decreased by half.

In fact, if we denote the partial sums of these series, respectively, by A_n and A'_n, then

$$A'_{3m}=\sum_{k=1}^{m}\left(\frac{1}{2k-1}-\frac{1}{4k-2}-\frac{1}{4k}\right)=\sum_{k=1}^{m}\left(\frac{1}{4k-2}-\frac{1}{4k}\right)$$

$$=\frac{1}{2}\sum_{k=1}^{m}\left(\frac{1}{2k-1}-\frac{1}{2k}\right)=\frac{1}{2}A_{2m},$$

so that $A'_{3m}\to(\log 2)/2$. Because

$$A'_{3m-1}=A'_{3m}+\frac{1}{4m} \quad \text{and} \quad A'_{3m-2}=A'_{3m-1}+\frac{1}{4m-2}$$

tend to the same limit $(\log 2)/2$, the series (4) converges and has as its limit precisely this number.

248. The multiplication of series. The term-by-term addition and subtraction of two convergent series, and similarly the term-by-term multiplication of a convergent series by a constant factor, have already been discussed in Sec. 235, (*3*) and (*4*). Now we will consider the question of the multiplication of series.

Let there be given two series

$$A=\sum_{n=1}^{\infty}a_n=a_1+a_2+\ldots+a_n+\ldots \tag{A}$$

and

$$B=\sum_{m=1}^{\infty}b_m=b_1+b_2+\ldots+b_m+\ldots. \tag{B}$$

Following the rule for the multiplication of finite sums, we consider here all possible pairwise products of the terms of these series: $a_i b_k$; the infinite rectangular matrix:

† Here it is only essential for our purposes that this sum differs from zero [Sec. 244, (6)].

$$\begin{array}{l} a_1b_1\ a_2b_1\ a_3b_1 \ldots a_ib_1 \ldots \\ a_1b_2\ a_2b_2\ a_3b_2 \ldots a_ib_2 \ldots \\ a_1b_3\ a_2b_3\ a_3b_3 \ldots a_ib_3 \ldots \\ \cdots\cdots\cdots\cdots \\ a_1b_k\ a_2b_k\ a_3b_k \ldots a_ib_k \ldots \\ \cdots\cdots\cdots\cdots \end{array} \tag{5}$$

is composed of just these products.

These products can be arranged in many ways in the form of simple sequences.

For instance, as was first done by Cauchy, one can write down the products by diagonals

$$\begin{array}{l} a_1b_1\ a_2b_1\ a_3b_1 \ldots \\ a_1b_2\ a_2b_2\ a_3b_2 \ldots \\ a_1b_3\ a_2b_3\ a_3b_3 \ldots \\ \cdots\cdots\cdots\cdots \end{array}$$

joining together the terms in one diagonal:

$$a_1b_1 + (a_1b_2 + a_2b_1) + (a_1b_3 + a_2b_2 + a_3b_1) + \ldots$$
$$+ (a_1b_n + a_2b_{n-1} + \ldots + a_{n-1}b_2 + a_nb_1) + \ldots. \tag{6}$$

CAUCHY'S THEOREM. *If both the series* (A) *and* (B) *converge absolutely, then the series* (6), *composed of the products* (5), *also converges and has as its sum the product of the sums AB.*

Proof. By hypothesis, the series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + \ldots + |a_n| + \ldots \tag{A*}$$

and

$$\sum_{m=1}^{\infty} |b_m| = |b_1| + |b_2| + \ldots + |b_m| + \ldots \tag{B*}$$

converge, i.e. they have finite sums, say, A^* and B^*. If for these series we form a series like (6):

$$|a_1|\cdot|b_1| + (|a_1|\cdot|b_2| + |a_2|\cdot|b_1|) + (|a_1|\cdot|b_3| + |a_2|\cdot|b_2| + |a_3|\cdot|b_1|) + \ldots + (|a_1|\cdot|b_n| + |a_2|\cdot|b_{n-1}| + \ldots + |a_{n-1}|\cdot|b_2| + |a_n|\cdot|b_1|) + \ldots, \qquad (6^*)$$

then it will be convergent, because its nth partial sum is less than

$$(|a_1| + |a_2| + \ldots + |a_n|)(|b_1| + |b_2| + \ldots + |b_n|) < A^*B^*$$

[Sec. 236]. Convergence is also preserved when the brackets in (6*) are omitted [see the remark in Sec. 245]. Hence it follows [Sec. 243] that the series obtained from (6) by omitting the brackets,

$$a_1b_1 + a_1b_2 + a_2b_1 + a_1b_3 + a_2b_2 + a_3b_1 + \ldots,$$

also converges and, in fact, converges absolutely. In such a case it possesses not only the associative, but also the permuting properties [Secs. 245, 246]. Its sum, coinciding with the sum of the series (6), can most simply of all be defined thus. We arrange its terms not by diagonals, but by squares:

$$\begin{array}{|c|c|c|c}
\hline
a_1b_1 & a_2b_1 & a_3b_1 & \ldots \\
\hline
a_1b_2 & a_2b_2 & a_3b_2 & \ldots \\
\hline
a_1b_3 & a_2b_3 & a_3b_3 & \ldots \\
\hline
\multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots\ldots} \\
\end{array}$$

and also we add together the successive groups which distinguish one square from another:

$$a_1b_1 + (a_1b_2 + a_2b_2 + a_2b_1) + (a_1b_3 + a_2b_3 + a_3b_3 + a_3b_2 + a_3b_1) + \ldots. \qquad (7)$$

The partial sums of the series (7) will be

$$A_1B_1, A_2B_2, A_3B_3, \ldots, A_nB_n, \ldots$$

(if, as usual, we denote by A_n, B_n the partial sums of the series (A) and (B)); they tend to the product AB, which is thus the sum not only of the series (7), but also of the series (6).

Examples. (1) Multiplying the series

$$\frac{1}{1-x} = \sum_{0}^{\infty} x^n = 1 + x + x^2 + \ldots + x^n + \ldots \qquad (|x| < 1)$$

by itself, we obtain:

$$\frac{1}{(1-x)^2} = \sum_{1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + \ldots + nx^{n-1} + \ldots .$$

(2) As we know already [Sec. 243, *(1)* (a)], the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

converges absolutely for all values of x. We denote its sum by $E(x)$. The product of $E(x)$ by $E(y)$ can be constructed by the rule for the multiplication of series. The general term of the product will be of the form:

$$1 \cdot \frac{y^n}{n!} + \frac{x}{1!} \cdot \frac{y^{n-1}}{(n-1)!} + \frac{x^2}{2!} \cdot \frac{y^{n-2}}{(n-2)!} + \ldots + \frac{x^k}{k!} \cdot \frac{y^{n-k}}{(n-k)!}$$

$$+ \ldots + \frac{x^n}{n!} \cdot 1 = \sum_{k=0}^{n} \frac{1}{k!(n-k)!} x^k y^{n-k}$$

$$= \frac{1}{n!} \sum_{k=0}^{n} C_n^k x^k y^{n-k} = \frac{(x+y)^n}{n!}.$$

Thus for the function $E(x)$ the relation

$$E(x) \cdot E(y) = E(x+y)$$

is obtained [later we shall see that $E(x) = e^x$].

(3) The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \ldots + (-1)^{n-1} \frac{1}{\sqrt{n}} + \ldots,$$

by Leibniz's theorem, converges, but *not absolutely* [Sec. 234, *(2)*]. If, nevertheless, we attempt to multiply it by itself by Cauchy's rule, then we obtain a series with the general term

$$(-1)^{n-1} \left(\frac{1}{1 \cdot \sqrt{n}} + \frac{1}{\sqrt{2} \cdot \sqrt{(n-1)}} + \ldots + \frac{1}{\sqrt{k}\,\sqrt{(n-k+1)}} + \ldots + \frac{1}{\sqrt{n} \cdot 1} \right).$$

Because each summand in the brackets is greater than $1/n$, then all the expressions are of absolute value > 1, and—in view of the violation of the necessary condition for convergence—the series diverges! This example is due to Cauchy himself.

Remark. One can show that, if of two convergent series (A) and (B) at least one converges absolutely, then Cauchy's rule still holds. But if both the series being multiplied together converge non-absolutely (as in the preceding example) their product will turn out to be divergent.

§ 5. Infinite products

249. Fundamental concepts. If

$$p_1, p_2, \cdots, p_n, \cdots \tag{1}$$

is some given sequence of numbers, then the symbol

$$p_1 \cdot p_2 \cdot p_3 \cdot \cdots \cdot p_n \cdot \cdots = \prod_{n=1}^{\infty} p_n \dagger \tag{2}$$

formed from them is called an *infinite product.*

We begin to multiply together, one after the other, the numbers (1), thus forming the partial products

$$P_1 = p_1, \; P_2 = p_1 \cdot p_2, \; P_3 = p_1 \cdot p_2 \cdot p_3, \; \cdots,$$
$$P_n = p_1 \cdot p_2 \cdot \cdots \cdot p_n, \ldots. \tag{3}$$

This sequence of partial products $\{P_n\}$ we shall frequently compare with the symbol (2).

The finite or infinite limit P of the partial products P_n as $n \to \infty$:

$$\lim P_n = P,$$

is called the value of the product (2), *and we write:*

$$P = p_1 \cdot p_2 \cdot \ldots \cdot p_n \cdot \ldots = \prod_{n=1}^{\infty} p_n.$$

If an infinite product has a finite value P, which, moreover, differs from zero, then the product itself is called convergent, and otherwise it is called divergent‡.

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† The symbol Π also denotes a product.

‡ Thus (we emphasize this), if $P = 0$, then as far as we are concerned the product will be divergent. Although this terminology is contrary to the terminology usual for series, yet it is current because it simplifies the formulation of many theorems.

The value of the whole product will be equal to zero if one of the factors of the product is zero. In our further investigations we will exclude this case, so that as far as we are concerned it always holds that $p_n \neq 0$.

The reader can easily establish an analogy with series [Sec. 234] and will see for himself that—similar to series—the investigation of infinite products is also only a special form of the study of a sequence and its limit. It is useful to become familiar with this form because in certain cases it is more convenient than other forms.

Examples. *(1)* $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$.

Because the partial products

$$P_n=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{1}{2}\cdot\frac{n+1}{n}\to\frac{1}{2},$$

the infinite product converges, and its value will be 1/2.

(2) Wallis's formula [Sec. 188]

$$\frac{\pi}{2}=\lim_{n\to\infty}\frac{2\cdot2\cdot4\cdot4\cdot\ldots\cdot2n\cdot2n}{1\cdot3\cdot3\cdot5\cdot\ldots\cdot(2n-1)\cdot(2n+1)}$$

clearly is equivalent to the expansion of the number $\pi/2$ in an infinite product

$$\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\ldots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdot\ldots.$$

It reduces to the formula

$$\prod_{m=1}^{\infty}\left[1-\frac{1}{(2m+1)^2}\right]=\frac{\pi}{4},\quad\prod_{m=1}^{\infty}\left(1-\frac{1}{4m^2}\right)=\frac{2}{\pi}.$$

250. The simplest theorems. The connection with series. Rejecting the first m terms in the infinite product (2), we obtain the residual product

$$\pi_m=p_{m+1}\cdot p_{m+2}\cdot\ldots\cdot p_{m+k}\cdot\ldots=\prod_{n=m+1}^{\infty}p_n, \tag{4}$$

which is entirely analogous to the remainder of an infinite series.

(1) If the product (2) *converges, then, for any m, the product* (4) *also converges: conversely, from the convergence of the product* (4) *follows the convergence of the original product* (2)†.

† We recall that we have presupposed once for all $p_n \neq 0$.

The proof we leave to the reader [cf. Sec. 235, *(1)*].

Thus, also in the case of an infinite product, *the rejection of a finite number of the first factors or the adjunction at the beginning of some new factors (distinct from zero) has no effect on its behaviour.*

(2) If the infinite product (2) *converges, then*

$$\lim_{m\to\infty} \pi_m = 1$$

[see (4)].

This follows from the equation

$$\pi_m = \frac{P}{P_m}$$

and from the fact that P_m tends to $P \neq 0$.

(3) If the infinite product (2) *converges, then*

$$\lim_{n\to\infty} p_n = 1.$$

In fact, together with P_n, P_{n-1} also tends to P, so that

$$\lim_{n\to\infty} p_n = \lim \frac{P_n}{P_{n-1}} = \frac{P}{P} = 1.$$

[Cf. Sec. 235, *(5)*.]

Without enumerating the other properties of infinite products analogous to the properties of infinite series, we turn to the connection established between the convergence of infinite products and series which enables us immediately to make use, for products, of the theory developed in detail for series.

In the case of convergent products the factors p_n, beginning at some point, will be positive [*(3)*]. However, in view of *(1)*, we shall suffer no loss of generality if, from now on, we presuppose that all $p_n > 0$.

(4) In order that the infinite product (2) *should converge, it is necessary and sufficient that the series*

$$\sum_{n=1}^{\infty} \log p_n \tag{5}$$

should converge.

When this condition is satisfied, if L is the sum of the series, we have:

$$P = e^L.$$

Denoting by L_n the partial sums of the series (5), we will have:

$$L_n = \log P_n, \qquad P_n = e^{L_n}.$$

From the continuity of the logarithmic and exponential functions, it now follows that if P_n tends to a finite positive limit P, then L_n tends to $\log P$, and conversely, if L_n has a finite limit L, then for P the limit will be e^L.

While investigating the convergence of an infinite product (2) it is often convenient, if we set

$$p_n = 1 + a_n,$$

to write it in the form

$$\prod_{n=1}^{\infty} (1 + a_n), \tag{2*}$$

and the series (5) in the form

$$\sum_{n=1}^{\infty} \log (1 + a_n). \tag{5*}$$

With this notation we have the simple theorem:

(5) If, for a sufficiently large n, it is true that

$$a_n > 0 \quad (\text{or } a_n < 0),$$

then the convergence of the series

$$\sum_{n=1}^{\infty} a_n \tag{6}$$

is necessary and sufficient for the convergence of the product (2*).

Because in every case it is necessary for the convergence both of the product (2*) and of the series (6) that

$$\lim_{n \to \infty} a_n = 0 \tag{7}$$

[see *(3)*], then we presuppose that this condition is satisfied. Then the relation

$$\lim_{n\to\infty} \frac{\log(1+a_n)}{a_n} = 1$$

holds [Sec. 65, *(1)*]. In such a case, in view of the fact that the terms of both the series (5*) and (6), beginning at some point, have a constant sign, then by Theorem 2, Sec. 237, these series converge or diverge together. Hence, in conjunction with *(4)*, our assertion follows.

We mention the case when an infinite product "diverges" to 0.

(6) In order that the infinite product (2) [*or* (2*)] *should have the value zero, it is necessary and sufficient that the series* (5) [*or* (5*)] *should have the sum* $-\infty$.

In particular, this will be true if $a_n < 0$ *and the series* (6) *diverges.*

We leave the proof to the reader.

In conclusion we make use of the connection between the product (2) [or (2*)] and the series (5) [or (5*)] to establish the concept of the absolute convergence of an infinite product. *A product is said to be absolutely convergent if and only if the corresponding series formed from the logarithms of its factors converges absolutely.*

The investigations of Sec. 246 and 247 enable one to conclude at once that absolutely convergent products possess the permuting property, while non-absolutely convergent products are known not to possess it.

It is easy to prove, by the method of *(5)*, that

(7) For the absolute convergence of the product (2*) *it is necessary and sufficient that the series* (6) *converges absolutely.*

251. Examples. *(1)* We will adapt some theorems, which have been proved, to infinite products:

(a) $\prod_{n=1}^{\infty} (1+1/n^x)$ converges when $x>1$ and diverges when $0<x\leqslant 1$, in accordance with the behaviour of the series $\sum_{n=1}^{\infty} 1/n^x$ [*(5)*].

(b) $\prod_{n=1}^{\infty} (1-1/n^x)$ converges when $x>1$ [*(5)*], and when $0<x\leqslant 1$ diverges to zero [*(6)*].

(2) We will prove that (when $\alpha > \beta > 0$)

$$\lim_{n\to\infty} \frac{\beta(\beta+1)\dots(\beta+n-1)}{\alpha(\alpha+1)\dots(\alpha+n-1)} = 0.$$

For this it is sufficient to establish the divergence of the infinite product

$$\prod_{n=0}^{\infty} \frac{\beta+n}{\alpha+n} \equiv \prod_{n=0}^{\infty} \left(1 - \frac{\alpha-\beta}{\alpha+n}\right)$$

or [see *(6)*] the divergence of the series

$$\sum_{n=0}^{\infty} \frac{\alpha-\beta}{\alpha+n}.$$

But this follows easily from comparison of the given series with the harmonic series.

This example shows how sometimes it is advantageous to reduce the search for the limit of a sequence to the investigation of an infinite product, with the use of the theory which has been developed for infinite products.

(3) Finally we mention a remarkable example of the transformation of an infinite product to a series, discovered by Euler. If we arrange the prime numbers in increasing order:

$$p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ \dots,\ p_k,\ \dots,$$

then when $x > 1$ the identity

$$\frac{1}{\left(1-\frac{1}{2^x}\right)\left(1-\frac{1}{3^x}\right)\left(1-\frac{1}{5^x}\right)\cdot\dots\cdot\left(1-\frac{1}{p_k^x}\right)\cdot\dots} = 1 + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \dots + \frac{1}{n^x} + \dots$$

or

$$\prod_{k=1}^{\infty} \frac{1}{1-\frac{1}{p_k^x}} = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

holds.

By the formula for the sum of a geometric progression we have:

$$\frac{1}{1-\frac{1}{p_k^x}} = 1 + \frac{1}{p_k^x} + \frac{1}{(p_k^2)^x} + \dots + \frac{1}{(p_k^m)^x} + \dots.$$

If we multiply together a finite number of such series, corresponding to all the prime numbers not exceeding the natural number N, then the partial product turns out to be equal to

$$P_x^{(N)} = \prod_{p_k \leqslant N} \frac{1}{1 - \frac{1}{p_k^x}} = \sum_{n=1}^{\infty}{}' \frac{1}{n^x} = \sum_{n=1}^{N} \frac{1}{n^x} + \sum_{n=N+1}^{\infty}{}' \frac{1}{n^x}, \qquad (8)$$

where the dash signifies that the summation is not extended over all the natural numbers, but only those (not counting unity) which in their factorization into prime factors contain only the prime numbers already introduced (the first N natural numbers, of course, have this property). Hence it is certainly true that

$$0 < P_x^{(N)} - \sum_{n=1}^{N} \frac{1}{n^x} < \sum_{n=N+1}^{\infty} \frac{1}{n^x}.$$

In view of the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^x}$ the expression on the right, being its remainder after the Nth term, tends to 0 as $N \to \infty$; taking limits, we obtain the required result.

This formula of Euler's was afterwards (1859) used by Riemann in his investigations into the distribution of prime numbers.

(4) When $x = 1$ the relation (8) still holds; hence

$$P_1^{(N)} = \prod_{p_k \leqslant N} \frac{1}{1 - \frac{1}{p_k}} > \sum_{n=1}^{N} \frac{1}{n} = H_N,$$

so that as $N \to \infty$, this time $P_1^{(N)} \to \infty$, i.e. the product

$$\frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\ldots\left(1-\frac{1}{p_k}\right)\ldots} = \prod_{k=1}^{\infty} \frac{1}{1-\frac{1}{p_k}}$$

diverges and has the value $+\infty$.

Herein lies the new proof given by Euler of the fact that the set of prime numbers is infinite (which, essentially, we have not used in the reasoning given); in fact, if this set were finite the product would also have a finite value. If we write down the result obtained thus:

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\ldots\left(1-\frac{1}{p_k}\right)\ldots = \prod_{k=1}^{\infty}\left(1-\frac{1}{p_k}\right) = 0,$$

then, using *(5)* we can deduce the divergence of the series

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{p_k}+\ldots=\sum_{k=1}^{\infty}\frac{1}{p_k}.$$

This important proposition gives, moreover, certain characteristics of the increase in the prime numbers; it is much stronger than the assertion on the divergence of the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$, because here we are only discussing part of its terms!

§ 6. The expansion of elementary functions in power series

252. Taylor series. We have already met in examples power series of the form:

$$\sum_{0}^{\infty}a_n x^n=a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots, \tag{1}$$

arranged in ascending powers of x.

We consider also power series of the more general form:

$$\sum_{0}^{\infty}a_n(x-x_0)^n=a_0+a_1(x-x_0)+\ldots+a_n(x-x_0)^n+\ldots, \tag{2}$$

arranged in ascending powers of the binomial $x-x_0$ (instead of x). Such a series does not differ essentially from a series of the form (1), because it is formed by a simple change of variable: $x-x_0=y$ (to within the notation for the variable).

In what follows we will study in detail the properties of power series, which in many ways resemble polynomials. Polynomials are segments of power series, which makes power series a convenient means for approximate calculation. In connection with all these properties, a very important question is the possibility of *expanding in advance a given function in powers of* $x-x_0$ (in particular, in powers of x), i.e. of representing it in the form of the sum of a series of type (2) [or (1)].

We shall here examine a similar expansion with respect to elementary functions, where the way to the solution of the question raised is opened for us by Taylor's formula, studied in detail in

Secs. 105–108. We presuppose that the function $f(x)$ considered has derivatives of all orders (thus it is continuous) in the interval $[x_0, x_0 + H]$ or $[x_0 - H, x_0](H > 0)$. Then, as we saw in Sec. 106, for all values of x in this interval the formula

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + r_n(x) \qquad (3)$$

holds, where the remainder $r_n(x)$ can be represented in one of the forms indicated in Sec. 106. Here we can take n as large as desired, i.e. we can extend this expansion to as high a power of $x - x_0$ as desired.

This, naturally, leads one to think about the infinite expansion:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \ldots, \qquad (4)$$

Such a series—independent of whether it converges and indeed has as its sum the function $f(x)$—is called the *Taylor series* for the function $f(x)$. It has the form (2), where its coefficients:

$$a_0 = f(x_0),\ a_1 = \frac{f'(x_0)}{1!},\ a_2 = \frac{f''(x_0)}{2!},\ \ldots,\ a_n = \frac{f^{(n)}(x_0)}{n!},\ \ldots$$

are called the *Taylor coefficients.*

Because the difference between $f(x)$ and the sum of $n+1$ terms of the Taylor series, in view of (3), is precisely $r_n(x)$, then, clearly: *in order that for some value of x the relation* (4) *should in fact hold, it is necessary and sufficient that the remainder* $r_n(x)$ *in the Taylor formula—for this value of x—should tend to* 0 *as n increases:*

$$\lim_{n \to \infty} r_n(x) = 0. \qquad (5)$$

In considering the problem of whether this equation holds, and for what particular values of x, we shall find it useful to have different forms for the remainder $r_n(x)$, which show its dependence on n.

On most occasions it is necessary to deal with the case when $x_0 = 0$ and the function $f(x)$ is expanded immediately in a series of powers of x:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots\,^{\dagger}; \quad (6)$$

this series has the form (1), with the coefficients:

$$a_0 = f(0),\ a_1 = \frac{f'(0)}{1!},\ a_2 = \frac{f''(0)}{2!},\ \ldots,\ a_n = \frac{f^{(n)}(0)}{n!},\ \ldots. \quad (7)$$

Now we write down the remainder $r_n(x)$ in detail, in a form suitable for this special case $x_0 = 0$ [Sec. 106]:

in Lagrange's form:

$$r_n(x) = \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}; \quad (8)$$

in Cauchy's form:

$$r_n(x) = \frac{f^{(n+1)}(\theta x)}{(n+1)!}(1-\theta)^n x^{n+1}. \quad (9)$$

Here, with regard to the factor θ, it is only known that it lies between 0 and 1, but it may be altered by a variation in x or n (and even in the transition from one form to the other).

We pass on to some concrete examples of expansions.

253. The expansion of the exponential and elementary trigonometrical functions in power series. We will first prove the following simple proposition, which at once includes a number of important cases:

If the function $f(x)$ has derivatives of all orders in the interval $[0, H]$ or $[-H, 0]$ $(H > 0)$, and all these derivatives for the values of x in the given interval are absolutely bounded by precisely one number:

$$|f^{(n)}(x)| \leqslant L \quad (10)$$

(where L does not depend on n), then in the whole interval the expansion (6) *holds.*

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† This series is usually called a *Maclaurin series.*

Indeed, taking the remainder $r_n(x)$ in Lagrange's form [see (8)], we have, by virtue of (10):

$$|r_n(x)| = \frac{|f^{(n+1)}(\theta x)|}{(n+1)!}|x|^{n+1} \leqslant L\frac{H^{n+1}}{(n+1)!}.$$

As n increases to infinity the expression $H^{n+1}/(n+1)!$ tends to 0 [Sec. 45, *(1)*]; this [by virtue of Sec. 235, *(5)*] also follows from the convergence of the series

$$1+\sum_{n=0}^{\infty}\frac{H^{n+1}}{(n+1)!}$$

[Sec. 239, *(2)*(a)]. But in such a case $r_n(x)$ also has the limit 0, which also proves our assertion.

This proposition is immediately applicable to the functions

$$f(x) = e^x,\ \sin x,\ \cos x$$

in any interval $[-H, H]$, because their derivatives, respectively equal to

$$f^{(n)}(x) = e^x,\ \sin(x+n\tfrac{1}{2}\pi),\ \cos(x+n\tfrac{1}{2}\pi),$$

will be bounded, as to their absolute magnitudes, by the number e^H for the function e^x, and by one for $\sin x$ and $\cos x$.

Because we have already written out the Taylor coefficients for these functions in Sec. 108, (1)–(3), we can immediately write out the expansions:

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ \dots\ +\frac{x^n}{n!}+\ \dots \tag{11}$$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ \dots\ +(-1)^{k-1}\frac{x^{2k-1}}{(2k-1)!}+\ \dots \tag{12}$$

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ \dots\ +(-1)^k\frac{x^{2k}}{2k!}+\ \dots. \tag{13}$$

All these hold—in view of the arbitrary nature of H—for any value of x.

254. Euler's formulae. The similar form of the coefficients of the remarkable expansions just obtained *suggests that there may be some connection between them* but—if we remain in the domain of real numbers—it is impossible to establish

such a connection. Euler succeeded in establishing one by introducing *powers with imaginary exponents.*

Although in the course we are presenting we are only considering real numbers and real variables, yet—deviating for the moment from this—we will quote here *Euler's formulae,* which express trigonometrical functions of real argument in terms of exponential functions of pure imaginary argument.

If we substitute (purely formally!) in (11), instead of the real number x, the imaginary number yi, then we obtain:

$$e^{yi} = 1 + yi + \frac{(yi)^2}{2!} + \frac{(yi)^3}{3!} + \frac{(yi)^4}{4!} + \frac{(yi)^5}{5!} + \dots$$

$$= 1 + yi - \frac{y^2}{2!} - \frac{y^3}{3!}i + \frac{y^4}{4!} + \frac{y^5}{5!}i + \dots$$

or, separating the real part from the imaginary,

$$e^{yi} = \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right).$$

The expressions in the parentheses are precisely the expansions of the functions $\cos y$ and $\sin y$ already known to us [see (13) and (12)]. Thus

$$e^{yi} = \cos y + i \sin y. \tag{14}$$

Replacing y by $-y$ here, we will have, analogously,

$$e^{-yi} = \cos y - i \sin y.$$

Term-by-term addition and subtraction of these two relations give us the familiar *Euler formulae*:

$$\cos y = \frac{e^{yi} + e^{-yi}}{2}, \quad \sin y = \frac{e^{yi} - e^{-yi}}{2i}. \tag{15}$$

The formulae (14) and (15) have wide application in analysis.

We turn to the logical meaning of what we have said†. We shall begin by considering the complex variable $z_n = x_n + iy_n$ depending on the natural index n. The definition of its limit is given in the same terms as for the real case [Sec. 28]: *the complex number $c = a + ib$ is said to be the limit of the variable z_n, if, whatever the value of the positive number ε, it is always possible to find a number N such that for $n > N$ the inequality $|z_n - c| < \varepsilon$ holds*‡.

Because

$$|z_n - c| = |(x_n - a) + i(y_n - b)| = \sqrt{[(x_n - a)^2 + (y_n - b)^2]},$$

then it is immediately clear that $z_n \to c$ *if and only if for the real and imaginary components x_n and y_n we have by definition $x_n \to a$ and $y_n \to b$.*

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† With the assumption that the reader is already familiar with complex numbers and complex variables from a course in higher algebra.

‡ Here $|\alpha + i\beta|$ denotes the "modulus" of the complex number $\alpha + i\beta$, equal to $\sqrt{(\alpha^2 + \beta^2)}$.

Now we consider the complex series

$$\sum_{n=1}^{\infty} c_n. \tag{C}$$

It is said to be *convergent*, if its partial sums

$$C_n = c_1 + c_2 + \ldots + c_n$$

as n increases tend to some complex number C as limit—*the sum of the series.* Expanding all the numbers mentioned here into their real and imaginary components:

$$C = A + iB, \quad c_n = a_n + ib_n, \quad C_n = A_n + iB_n,$$

where

$$A_n = a_1 + a_2 + \ldots + a_n, \quad B_n = b_1 + b_2 + \ldots + b_n,$$

from the preceding remarks we conclude that

$$C_n \to C \text{ at the same time as } A_n \to A,\ B_n \to B,$$

i.e. the convergence of the complex series (C) to the sum C is equivalent to the convergence, separately, of the real series

$$\sum_{n=1}^{\infty} a_n \quad \text{(A)} \quad \text{and} \quad \sum_{n=1}^{\infty} b_n \quad \text{(B)},$$

respectively, to the sums A and B. Hence, in particular, it follows that convergent complex series—like convergent real series—possess the associative property [Sec. 245].

If the series $\sum_{1}^{\infty} |c_n|$, formed from the moduli of the series (C), converges, then, in view of the inequality

$$|a_n| \leqslant |c_n|, \quad |b_n| \leqslant |c_n|,$$

the series

$$\sum_{1}^{\infty} |a_n|, \quad \sum_{1}^{\infty} |b_n|,$$

also converges, hence follows the convergence of the series (A) and (B) [Sec. 243], and consequently of the series (C). In this case the series (C) is said to be *absolutely convergent*. For instance, the series

$$1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots \tag{16}$$

for any complex value of z converges absolutely, because the real series of the moduli of its terms

$$1 + \frac{|z|}{1!} + \frac{|z|^2}{2!} + \ldots + \frac{|z|^n}{n!} + \ldots$$

converges.

An absolutely convergent series (C) possesses the permuting property, because the series (A) and (B) possess this property. Finally, the theorem on multiplication

can be extended to complex absolutely convergent series [Sec. 248]; the proof given for the real case—after making a few remarks—can be repeated here verbatim.

Now we raise the question of defining in a general form the power e^z with complex z. In the case when z is a real number, x, this power has already been defined; in the preceding section we proved that for it the relation (11) holds. In the case of an imaginary exponent z the power e^z has not yet been defined; copying the expansion given, on this occasion we put e^z equal to the sum of the series (16) by definition (it is known to exist!). It is very important that here *the fundamental property of an exponential function*

$$e^z \cdot e^{z'} = e^{z+z'}$$

is preserved; one can easily verify this by multiplying together the series giving e^z and $e^{z'}$ [cf. Sec. 248, *(2)*].

Thus, when above we replaced x by yi in the expansion (11), we simply made use of the given extension of the concept of a power.

We note in conclusion that, if $z = x + yi$, then by the rule for exponents $e^z = e^x \cdot e^{yi}$ so that, using (14), finally:

$$e^{x+yi} = e^x(\cos y + i \sin y). \tag{17}$$

255. The expansion for the inverse tangent. The general proposition proved above [Sec. 253] is not applicable to the function $y = \arctan x$. In fact, the expression for its nth derivative, found in Sec. 96, *(5)*:

$$y^{(n)} = (n-1)! \cos^n y \cdot \sin n(y + \tfrac{1}{2}\pi) \tag{18}$$

does not guarantee the existence of a general bound for all $y^{(n)}$!

Because the corresponding Taylor series [see Sec. 108, *(6)*]

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^{k-1} \frac{x^{2k-1}}{2k-1} + \ldots$$

converges only in the interval $[-1, 1]$†, then, outside this interval, it is not appropriate to speak of the expression of the function $\arctan x$ in terms of this series. On the contrary, for $|x| \leqslant 1$ by Lagrange's formula (8) [using (18) and setting $y_\theta = \arctan \theta x$] we have

$$|r_n(x)| \leqslant \frac{\left|\cos^{n+1} y_\theta \cdot \sin(n+1)\left(y_\theta + \frac{\pi}{2}\right)\right|}{n+1} |x|^{n+1} \leqslant \frac{1}{n+1}.$$

† By d'Alembert's test [Sec. 243] it is easy to ascertain that the series converges (absolutely), if $|x| < 1$, and diverges when $|x| > 1$. The convergence (non-absolute) when $x = \pm 1$ follows from Leibniz's theorem [Sec. 244].

Hence it is clear that $r_n(x) \to 0$, so that for all values of x in the interval $[-1, 1]$ the expansion

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^{k-1}\frac{x^{2k-1}}{2k-1} + \ldots \quad (19)$$

holds.

We have already emphasized once that, although arc $\tan x$ also has a definite meaning outside this interval, the expansion (19) does not exist there, in so far as the series does not have a sum.

From series (19), with $x = 1$, we obtain, in particular, the well-known *Leibniz series*

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots + (-1)^{k-1}\frac{1}{2k-1} + \ldots, \quad (20)$$

the first series, in the history of analysis, to represent the expansion of the number π.

256. Logarithmic series. If as the function $f(x)$ we take $\log(1+x)$ $(x > -1)$, then the corresponding Taylor series will be [Sec. 108, *(5)*]:

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots.$$

It converges only for values of x in the interval $(-1, 1]$†; this means that only for these values does the investigation of the behaviour of the remainder $r_n(x)$ have any meaning.

Let us take it first in Lagrange's form (8). Because

$$f^{(n+1)}(x) = (-1)^n \frac{n!}{(1+x)^{n+1}}$$

[Sec. 96, *(2)*], then

$$r_n(x) = (-1)^n \frac{1}{n+1}\frac{x^{n+1}}{(1+\theta x)^{n+1}} \qquad (0 < \theta < 1).$$

If $0 \leqslant x \leqslant 1$, then the last factor does not exceed 1 and hence $|r_n(x)| \leqslant 1/(n+1)$, so that $r_n(x) \to 0$ $(\text{as } n \to \infty)$.

† Here also absolute convergence when $|x| < 1$ and divergence when $|x| > 1$ are established with the help of d'Alembert's test. When $x = 1$, (non-absolute) convergence occurs, by Leibniz's theorem, and when $x = -1$, we obtain the divergent harmonic series by a change of signs.

But when $x<0$ the behaviour of this factor becomes indefinable, and it is appropriate to have recourse to Cauchy's form of the remainder [see (9)].

We have

$$r_n(x) = (-1)^n x^{n+1} \frac{(1-\theta)^n}{(1+\theta x)^{n+1}} \qquad (0<\theta<1),$$

so that

$$|r_n(x)| \leqslant \frac{|x|^{n+1}}{1-|x|} \left(\frac{1-\theta}{1+\theta x}\right)^n.$$

Since, when $x>-1$, it will be true that $1+\theta x>1-\theta$, the last factor is less than 1; consequently, only when $|x|<1$ is it known that $r_n(x)\to 0$.

It is interesting that, although Cauchy's form entirely exhausts the question for all values of x between -1 and 1, it gives us nothing when $x=1$; in this case we obtain

$$|r_n(1)|<(1-\theta)^n,$$

but in view of the possibility of changing θ together with n, it is impossible to deduce that $(1-\theta)^n\to 0$.

Thus, taking all our results together, for all values of x in the interval $(-1, 1]$, it will, in fact, be true that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots. \qquad (21)$$

In particular, when $x=1$ we obtain the series already known to us:

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \ldots + (-1)^{n-1}\frac{1}{n} + \ldots. \qquad (22)$$

From the series (21) it is possible to derive other useful expansions. For instance, replacing x by $-x$ in it (here we assume $|x|<1$) and subtracting the series obtained term by term from the series (21), we arrive at the following series:

$$\log\frac{1+x}{1-x} = 2x\left(1 + \frac{1}{3}x^2 + \frac{1}{5}x^4 + \ldots + \frac{1}{2m+1}x^{2m} + \ldots\right). \qquad (23)$$

Here we set $x = 1/(2n+1)$, where n is an arbitrary natural number. Because then

$$\frac{1+x}{1-x} = \frac{n+1}{n},$$

we obtain the expansion

$$\log\frac{n+1}{n} = \frac{2}{2n+1}\left[1 + \frac{1}{3}\frac{1}{(2n+1)^2} + \frac{1}{5}\frac{1}{(2n+1)^4} + \dots\right], \qquad (24)$$

which we shall need in computing logarithms [Sec. 262].

257. Stirling's formula. As another application of the expansion (24) we shall prove how with the help of it one can derive an important formula of analysis, known as Stirling's formula†.

For this purpose we write out the expansion (24) in the form:

$$\left(n+\frac{1}{2}\right)\log\left(1+\frac{1}{n}\right) = 1 + \frac{1}{3}\frac{1}{(2n+1)^2} + \frac{1}{5}\frac{1}{(2n+1)^4} + \dots.$$

This expression, clearly, is greater than 1, but less than

$$1 + \frac{1}{3}\left[\frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^4} + \dots\right] = 1 + \frac{1}{12n(n+1)}.$$

Thus we have

$$1 < \left(n+\frac{1}{2}\right)\log\left(1+\frac{1}{n}\right) < 1 + \frac{1}{12n(n+1)};$$

hence, taking potentials, we find

$$e < \left(1+\frac{1}{n}\right)^{n+1/2} < e^{1+\frac{1}{12n(n+1)}}.$$

Now we introduce a function with a natural index n:

$$a_n = \frac{n!\, e^n}{n^{n+1/2}}.$$

Then

$$\frac{a_n}{a_{n+1}} = \frac{\left(1+\frac{1}{n}\right)^{n+1/2}}{e},$$

† James Stirling (1692–1770) was a British mathematician. The formula quoted below was published by him in 1730.

and from the preceding inequality it follows that

$$1 < \frac{a_n}{a_{n+1}} < e^{1/12n(n+1)} = \frac{e^{1/12n}}{e^{1/12(n+1)}}$$

so that, on the one hand, $a_n > a_{n+1}$, but on the other,

$$a_n e^{-1/12n} < a_{n+1} e^{-1/12(n+1)}.$$

Thus, as n increases the variable a_n decreases (remaining bounded below, for instance, by zero), and tends to a finite limit a, but the variable $a_n e^{-1/12n}$ increases, tending, clearly, to the same limit a (because $e^{-1/12n} \to 1$). Because, for any n, the inequality

$$a_n e^{-1/12n} < a < a_n$$

is satisfied, then it is possible to find a number θ, contained between zero and 1, such that

$$a = a_n e^{-\theta/12n} \quad \text{or} \quad a_n = a e^{\theta/12n}.$$

(We note that the number θ, generally speaking, depends on n.) Bearing in mind the definition of the variable a_n, we find:

$$n! = a\sqrt{n}\left(\frac{n}{e}\right)^n e^{\theta/12n} \qquad (0 < \theta < 1). \tag{25}$$

Now it remains to define the magnitude of the constant a. For this purpose we call to mind Wallis's formula [Sec. 188], which can be written in the form:

$$\sqrt{\frac{\pi}{2}} = \lim_{n\to\infty} \frac{1}{\sqrt{(2n+1)}} \frac{2n!!}{(2n-1)!!}.$$

The last factor can be transformed in the following way:

$$\frac{2n!!}{(2n-1)!!} = \frac{(2n!!)^2}{2n!} = \frac{2^{2n}(n!)^2}{2n!};$$

substituting here instead of $n!$ the expression

$$\sqrt{n}\left(\frac{n}{e}\right)^n a_n,$$

and instead of $2n!$ the analogous expression

$$\sqrt{(2n)}\left(\frac{2n}{e}\right)^{2n} a_{2n},$$

after some elementary simplification we obtain

$$\sqrt{\frac{\pi}{2}} = \lim_{n\to\infty} \frac{n}{\sqrt{[2n(2n+1)]}} \frac{a_n^2}{a_{2n}} = \frac{a}{2},$$

so that $a = \sqrt{(2\pi)}$.

Substituting this value for a in the formula (25), we arrive at *Stirling's formula*

$$n! = \sqrt{(2\pi n)}\left(\frac{n}{e}\right)^n e^{\theta/12n} \quad (0<\theta<1).$$

If we reject the last factor, then we obtain the approximate formula†

$$n! \doteq \sqrt{(2\pi n)}\left(\frac{n}{e}\right)^n,$$

which enables one to calculate the value of the factorial $n!$ for large values of n. The factor rejected enables one to estimate the relative error allowed, which is clearly less than $e^{1/12n}-1$.

Stirling's formula is often applicable in the theory of probability and statistics.

258. Binomial series. Finally we shall take $f(x)=(1+x)^m$, where m is any real number, distinct from 0 and all natural numbers (for natural m the well-known finite expansion is obtained by Newton's formula). In this case the Taylor series has the form [Sec. 108, *(4)*]:

$$1+mx+\frac{m(m-1)}{1\cdot 2}x^2+\ldots + \frac{m(m-1)\ldots(m-n+1)}{1\cdot 2\cdot\ldots\cdot n}x^n+\ldots;$$

it is called the *binomial series*, and its coefficients the *binomial coefficients*. With the hypotheses made with respect to m, not one of these coefficients will be zero (on the contrary, if m were a natural number, then the coefficient for x^{m+1} and all those following it would turn out to be zero). With the help of d'Alembert's test [Sec. 243] it is easy to establish that when $|x|<1$ the binomial series converges absolutely, and when $|x|>1$ it diverges. The investigation of the remainder $r_n(x)$ we will carry out under the hypothesis that $|x|<1$, and moreover we shall take it at once in Cauchy's form (9) (Lagrange's form does not provide the answer here for all the values of x which interest us).

Because

$$f^{(n+1)}(x)=m(m-1)\ldots(m-n+1)(m-n)(1+x)^{m-n-1},$$

then the remainder is

$$r_n(x)=\frac{m(m-1)\ldots(m-n)(1+\theta x)^{m-n-1}}{1\cdot 2\ldots n}(1-\theta)^n x^{n+1}.$$

† This, strictly speaking, was also established by Stirling.

By regrouping its factors, we represent it in the form:

$$r_n(x) = \frac{(m-1)(m-2)\dots\overline{(m-1-n+1)}}{1\cdot 2\dots n}x^n \times$$

$$\times\, mx(1+\theta x)^{m-1}\left(\frac{1-\theta}{1+\theta x}\right)^n.$$

The first of these three expressions is the general term of the binomial series, but corresponding to the exponent $m-1$; because when $|x|<1$ the binomial series converges, whatever may be the exponent, then this expression tends to zero as $n\to\infty$. As far as the other two expressions are concerned, the second, in its absolute magnitude, lies between the bounds

$$|mx|\cdot(1-|x|)^{m-1} \quad \text{and} \quad |mx|\cdot(1+|x|)^{m-1},$$

not depending on n, and the third, as in Sec. 256, is less than 1. Thus, $r_n(x)\to 0$, i.e. for $|x|<1$ the expression

$$(1+x)^m = 1+mx+\frac{m(m-1)}{1\cdot 2}x^2+\dots$$

$$+\frac{m(m-1)\dots(m-n+1)}{1\cdot 2\dots n}x^n+\dots \qquad (26)$$

holds, and is also connected with Newton's name.

We have not considered the question of its validity for the values $x=\pm 1$, because its solution requires a painstaking investigation of the remainder. We will limit ourselves to remarking that when $x=1$ the expression (26) holds for $m>-1$, and when $x=-1$ for $m>0$.

We will note some special cases of the binomial series, corresponding, for instance, to $m=-1, 1/2, -1/2$:

$$\frac{1}{1+x} = 1-x+x^2-\dots+(-1)^{n-1}x^n+\dots \qquad (-1<x<1)$$

(the usual geometric progression), then,

$$\sqrt{(1+x)} = 1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4+\dots$$

$$+(-1)^{n-1}\frac{(2n-3)!!}{2n!!}x^n+\dots \qquad (-1\leqslant x\leqslant 1)$$

and

$$\frac{1}{\sqrt{(1+x)}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4 - \ldots$$
$$+ (-1)^n \frac{(2n-1)!!}{2n!!}x^n + \ldots \quad (-1 < x \leqslant 1).$$

It is important to emphasize that in the case of rational m the sum of the binomial series always gives an arithmetical value of the radical.

259. A remark on the study of the remainder. In the examples considered above of the expansion of a function in a Taylor series it always turned out that for all the values of x for which the series converged, its sum was precisely equal to the original function. Therefore the reader may have begun to suspect that in general it is sufficient to establish the convergence of the series, without even verifying the relation (5):

$$\lim_{n\to\infty} r_n(x) = 0,$$

in order to guarantee the expansion (4) or (6) (pp. 45–46).

In fact, however, this is not so. Following Cauchy, we consider, by way of example, the function defined by the equations

$$f(x) = e^{-1/x^2} \quad (\text{when } x \neq 0), \quad f(0) = 0.$$

When $x \neq 0$, it has derivatives of all orders

$$f'(x) = \frac{2}{x^3}e^{-1/x^2}, \quad f''(x) = \left(-\frac{6}{x^4} + \frac{4}{x^6}\right)e^{-1/x^2}, \ldots$$

and, in general,

$$f^{(n)}(x) = P_n\left(\frac{1}{x}\right)e^{-1/x^2} \quad (n = 1, 2, 3, \ldots), \tag{27}$$

where $P_n(z)$ is a polynomial of z (of degree $3n$). It is easy to ascertain the generality of this rule by the method of mathematical induction.

Now we will establish that also at the point $x = 0$ there exist derivatives of all orders for our function, where they are all equal to zero†. In fact, first of all,

$$\frac{f(x) - f(0)}{x} = \frac{\frac{1}{x}}{e^{1/x^2}} \to 0 \quad \text{as} \quad x \to 0,$$

† We note, by the way, that Lagrange considered that it would be impossible for a function (not identically equal to zero) to vanish at any point with all its derivatives. Cauchy constructed his example to refute this opinion of Lagrange's.

so that $f'(0) = 0$†. Let us suppose that the assertion has been proved correct for all the derivatives up to the nth order inclusive. Then [see (27)]

$$\frac{f^{(n)}(x)-f^{(n)}(0)}{x} = \frac{\frac{1}{x}P_n\left(\frac{1}{x}\right)}{e^{1/x^2}} \to 0 \quad \text{as} \quad x \to 0,$$

in so far as the numerator is the sum of terms of the form c/x^m. This means that $f^{(n+1)}(0) = 0$ also. By the method of mathematical induction the assertion is proved.

The Maclaurin series for this function—because its coefficients are, without exception, zero—of course converges for all values of x, but not one of these values (except $x = 0$) reproduces the original function.

Thus the preliminary definition of the domain of the convergence of a Taylor series only gives a partial definition: it sometimes enables us to exclude in advance from our investigation those values of x for which a negative result is certain beforehand, in so far as the series diverges. For those points where the series converges, only examination of the remainder of the Taylor formula makes it possible to prove that the sum of the Taylor series will in fact be the original function.

§ 7. Approximate calculations using series

260. Statement of the problem. With the example of the expansions actually obtained by us we will show how infinite series can be used for the purposes of approximate calculations. We shall begin with a number of general remarks.

If the number A is expanded as a series

$$A = a_1 + a_2 + a_3 + \ldots + a_n + \ldots,$$

where $a_1, a_2, a_3, \ldots$ are convenient (usually rational) numbers, and we set approximately:

$$A \doteq A_n = a_1 + a_2 + \ldots + a_n,$$

then the correction due to the rejection of all the remaining terms is expressed by the remainder

$$\alpha_n = a_{n+1} + a_{n+2} + \ldots.$$

For sufficiently large n this error becomes as small as desired, so that A_n reproduces A to any degree of accuracy determined in advance.

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† This follows from the general remark (which it is necessary to have in view also in what follows) that, whatever the value of the exponent $\mu > 0$,

$$\lim_{z \to \infty} \frac{z^\mu}{e^z} = 0$$

[see Sec. 121, *(4)*]. Here the part of z is played by $1/x^2$ (as $x \to 0$).

We are interested in the possibility of making a simple approximation for the remainder α_n; we could do this, and basically it amounts to calculating successive partial sums, when an approximation of the required degree of accuracy will be obtained.

If the series considered turns out to be an alternating series and moreover is monotonically decreasing in the absolute values of its terms, (of the "Leibniz type"), then, as we have seen [Sec. 244, Remark], the remainder has the sign of its first term and is less than it in absolute value. This approximation leaves nothing to be desired as far as simplicity is concerned.

The situation is somewhat more complicated in the case of a positive series. Then usually one tries to find a positive series with larger terms, which can be summed: $\sum_1^\infty a_n'$, $a_n' \geqslant a_n$, and as an approximation for the remainder α_n we take the value of the remainder α_n' of this new series: $\alpha_n \leqslant \alpha_n'$. For instance, for the series $\sum_1^\infty \frac{1}{m^2}$ we obtain

$$\sum_{m=n+1}^{\infty} \frac{1}{m^2} < \sum_{m=n+1}^{\infty} \frac{1}{m(m-1)} = \sum_{m=n+1}^{\infty} \left(\frac{1}{m-1} - \frac{1}{m}\right) = \frac{1}{n},$$

and for the series $1 + \sum_1^\infty \frac{1}{m!}$:

$$\sum_{m=n+1}^{\infty} \frac{1}{m!} = \frac{1}{n!} \sum_{m=n+1}^{\infty} \frac{1}{(n+1)\ldots m} < \frac{1}{n!} \sum_{m=n+1}^{\infty} \frac{1}{(n+1)^{m-n}} = \frac{1}{n!\,n}$$

(we in fact used this approximation in calculating the number e in Sec. 49).

Usually a decimal approximation for the number A is sought when the terms of the series cannot be expressed as decimal fractions. The rounding off, in converting them to decimal fractions, serves as a new source of error, which must also be taken into account.

Finally, we note that not every series, which has as its sum the number A which interests us, is of use in the actual approximation for this number (even if its terms are simple, and the evaluation of its remainder can be easily performed). The problem lies in the speed of the convergence, i.e. in the speed of the approximation of the partial sum to the number A.

We will take as an example the series [see Sec. 255, (20) and Sec. 256, (22)]

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \quad \text{and} \quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots,$$

which give, respectively, the expansions of the number $\pi/4$ and $\log 2$. They converge very slowly, and in order to find, with their help, approximate values for these

numbers to a high degree of accuracy, it would be necessary to add together an immense number of terms. Below we will find, without a great deal of trouble, decimal approximations for the given numbers to a high degree of accuracy, by using more suitable series.

261. The calculation of the number π. We make use of the known series for the inverse tangent [Sec. 255, (19)]:

$$\operatorname{arc\,tan} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \quad (-1 \leqslant x \leqslant 1).$$

If we take $x = 1/\sqrt{3}$, then $\operatorname{arc\,tan} x = \pi/6$, and we obtain the series

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{5}\cdot\frac{1}{3^2} - \frac{1}{7}\cdot\frac{1}{3^3} + \ldots\right),$$

which is suitable for the calculation.

There exist, however, much more convenient series for the calculation of the number π. We set $\alpha = \operatorname{arc\,tan}(1/5)$, then

$$\tan\alpha = \frac{1}{5}, \quad \tan 2\alpha = \frac{\frac{2}{5}}{1 - \frac{1}{25}} = \frac{5}{12}, \quad \tan 4\alpha = \frac{\frac{10}{12}}{1 - \frac{25}{144}} = \frac{120}{119}.$$

In view of the proximity of this number to 1, it is clear that the angle 4α is near $\pi/4$; setting $\beta = 4\alpha - (\pi/4)$, we will have:

$$\tan\beta = \frac{\frac{120}{119} - 1}{1 + \frac{120}{119}} = \frac{1}{239}, \quad \text{hence} \quad \beta = \operatorname{arc\,tan}\frac{1}{239}.$$

Hence we have obtained the formula:

$$\pi = 16\alpha - 4\beta = 16\left\{\frac{1}{5} - \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{5}\cdot\frac{1}{5^5} - \frac{1}{7}\cdot\frac{1}{5^7} + \frac{1}{9}\cdot\frac{1}{5^9} - \frac{1}{11}\cdot\frac{1}{5^{11}} + \ldots\right\} - 4\left\{\frac{1}{239} - \frac{1}{3}\cdot\frac{1}{239^3} + \ldots\right\}.$$

Using this we will calculate the number π to seven decimal figures. For this purpose the terms of the formula which have actually been written down will be sufficient. Because both series are of the Leibniz type, the errors in the minuend and subtrahend, due to the rejection of the terms which have not been written down, will be, respectively:

$$0 < \Delta_1 < \frac{16}{13\cdot 5^{13}} < \frac{1}{10^9} \quad \text{and} \quad 0 < \Delta_2 < \frac{4}{5\cdot 239^5} < \frac{1}{10^9}.$$

The retained terms we turn into decimal fractions, rounding them off (by the rule for the complement) at the eighth decimal place. If we reduce the calculation to a table (+ or − in the brackets indicates the sign of the correction):

$$\frac{16}{5} = 3.20000000 \qquad \frac{16}{3\cdot 5^3} = 0.04266667(-)$$

$$+\frac{16}{5\cdot 5^5} = 0.00102400 \qquad +\frac{16}{7\cdot 5^5} = 0.00002926(-)$$

$$\frac{16}{9\cdot 5^9} = 0.00000091(+) \qquad \frac{16}{11\cdot 5^{11}} = 0.00000003(-)$$

$$3.20102491 \qquad 0.04269596$$

$$3.20102491 - 0.04269596 = 3.15832895$$

$$\frac{4}{239} = 0.01673640(+)$$

$$-\frac{4}{3\cdot 239^3} = 0.00000010(-)$$

$$0.01673630$$

Taking into account all the corrections, we have:

$$3.15832895 < 16\alpha < 3.15832898$$

$$-0.01673632 < -4\beta < -0.01673630,$$

so that

$$3.14159263 < \pi < 3.14159268.$$

Thus, finally, $\pi = 3.1415926\ldots$, where all the figures written down are correct.

262. The calculation of logarithms. At the basis of the calculation lies the series (24), Sec. 256:

$$\log\frac{n+1}{n} = \log(n+1) - \log n$$

$$= \frac{2}{2n+1}\left[1 + \frac{1}{3}\cdot\frac{1}{(2n+1)^2} + \frac{1}{5}\cdot\frac{1}{(2n+1)^4} + \ldots\right]. \qquad (1)$$

When $n = 1$ we obtain the expansion for log 2:

$$\log 2 = \frac{2}{3}\left(1 + \frac{1}{3}\cdot\frac{1}{9} + \frac{1}{5}\cdot\frac{1}{9^2} + \frac{1}{7}\cdot\frac{1}{9^3} + \frac{1}{9}\cdot\frac{1}{9^4} + \frac{1}{11}\cdot\frac{1}{9^5} + \frac{1}{13}\cdot\frac{1}{9^6} + \frac{1}{15}\cdot\frac{1}{9^7} + \frac{1}{17}\cdot\frac{1}{9^8} + \ldots\right).$$

This series is perfectly suitable for the calculation. We will prove, for instance, that, by confining oneself merely to the terms written down, one can find log 2 to nine correct decimal figures.

In fact, if we reject the terms of this series beginning with the tenth, then the corresponding error will be:

$$\Delta = \frac{2}{3}\left(\frac{1}{19}\cdot\frac{1}{9^9} + \frac{1}{21}\cdot\frac{1}{9^{10}} + \ldots\right) < \frac{2}{3\cdot 19\cdot 9^9}\left(1 + \frac{1}{9} + \frac{1}{9^2} + \ldots\right)$$

$$= \frac{1}{12\cdot 19\cdot 9^8} < \frac{2}{10^{10}}.$$

The calculation, to 10 decimal figures, reduced to a table is:

$$\frac{2}{3} = 0.66666\ 66667\ (-)$$

$$\frac{2}{3\cdot 3\cdot 9} = 0.02469\ 13580\ (+)$$

$$\frac{2}{3\cdot 5\cdot 9^2} = 0.00164\ 60905\ (+)$$

$$\frac{2}{3\cdot 7\cdot 9^3} = 0.00013\ 06421\ (+)$$

$$\frac{2}{3\cdot 9\cdot 9^4} = 0.00001\ 12901\ (-)$$

$$\frac{2}{3\cdot 11\cdot 9^5} = 0.00000\ 10264\ (-)$$

$$\frac{2}{3\cdot 13\cdot 9^6} = 0.00000\ 00965\ (-)$$

$$\frac{2}{3\cdot 15\cdot 9^7} = 0.00000\ 00093\ (-)$$

$$\frac{2}{3\cdot 17\cdot 9^8} = 0.00000\ 00009\ (+)$$

$$0.69314\ 71805$$

Taking into account all the errors, we have:

$$0.6931471802 < \log 2 < 0.6931471809,$$

so that

$$\log 2 = 0.693147180\ldots,$$

and all nine figures given are correct.

Now letting $n = 4$ in (1) we find:

$$\log 5 = 2\log 2 + \frac{2}{9}\left(1 + \frac{1}{3}\cdot\frac{1}{81} + \frac{1}{5}\cdot\frac{1}{81^2} + \ldots\right).$$

By making use of the value for $\log 2$ already calculated, it is easy to calculate $\log 5$ with this formula, and then $\log 10 = \log 2 + \log 5$. After this, with an arbitrary degree of accuracy, one can calculate the module

$$M = \frac{1}{\log 10}$$

for changing natural logarithms to logarithms to the base ten; it is equal to $M = 0.434294481\ldots$. By multiplying by this module, we can find the logarithms to the base ten of 2 and 5.

Going over to logarithms to the base ten in the basic formula (1):

$$\log_{10}(n+1) - \log_{10} n = \frac{2M}{2n+1}\left[1 + \frac{1}{3}\cdot\frac{1}{(2n+1)^2} + \right.$$

$$\left. + \frac{1}{5}\cdot\frac{1}{(2n+1)^4} + \ldots\right]. \qquad (2)$$

Letting $n = 80 = 2^3 \cdot 10$ and taking into account that $n+1 = 81 = 3^4$, we find

$$4\log_{10} 3 - 3\log_{10} 2 = \frac{2M}{161}\left(1 + \frac{1}{3}\cdot\frac{1}{25921} + \frac{1}{5}\cdot\frac{1}{25921^2} + \ldots\right) + 1,$$

hence it is easy to find $\log_{10} 3$. Further, letting $n = 2400 = 3\cdot 2^3\cdot 10^2$ in the formula (2), we will have $n+1 = 2401 = 7^4$ and

$$4\log_{10} 7 - 3\log_{10} 2 - \log_{10} 3 - 2 = \frac{2M}{4801}\left(1 + \frac{1}{3}\cdot\frac{1}{23049601} + \ldots\right),$$

so that we also find $\log_{10} 7$. By selecting similar numerical combinations, we can find, with arbitrary degree of accuracy, the logarithms of prime numbers, and with them, by means of multiplication by natural factors and addition, one can find the logarithms of composite numbers.

It would be possible to work in another way, by calculating immediately the logarithms of successive natural numbers and going over from $\log_{10} n$ to $\log_{10}(n+1)$ with the help of formula (2). Thus, for the calculation of the logarithms of the numbers from 1000 to 10,000, let us take only one term in formula (2), i.e. we set approximately

$$\log_{10}(n+1) - \log_{10} n = \frac{2M}{2n+1} \qquad (10^3 \leqslant n \leqslant 10^4).$$

The error here will be

$$\begin{aligned}\Delta &= \frac{2M}{2n+1}\left[\frac{1}{3}\cdot\frac{1}{(2n+1)^2}+\frac{1}{5}\cdot\frac{1}{(2n+1)^4}+\ldots\right]\\ &< \frac{2M}{3(2n+1)^3}\left[1+\frac{1}{(2n+1)^2}+\frac{1}{(2n+1)^4}+\ldots\right]\\ &= \frac{2M}{3(2n+1)\cdot 2n\cdot(2n+2)} < \frac{2M}{24n^3}.\end{aligned}$$

Because we have $n \geqslant 10^3$, and $2M < 1$, then

$$\Delta < \frac{1}{24\cdot 10^9} < \frac{1}{2\cdot 10^{10}}.$$

Even if all the corrections are added together, then in general the whole error would be less than $10^4/(2\cdot 10^{10}) = 1/(2\cdot 10^6)$. But it is possible to avoid such an accumulation of errors by writing down the whole series of control logarithms by the first method. By this means one can achieve a considerable degree of accuracy, while keeping at the same time the present second method as an automatic calculation, which is very valuable, particularly in making up large tables.

CHAPTER 16

SEQUENCES AND SERIES OF FUNCTIONS

§ 1. Uniform convergence

263. Introductory remarks. We have studied above infinite sequences and their limits, infinite series and their sums; the elements of the sequences or the terms of the series were constant numbers. It is true that sometimes real variables of one sort or another occurred in them as parameters, but when they were being studied definite constant values were ascribed to them. Thus, for instance, when we established that the sequence

$$\left(1+\frac{x}{1}\right)^1, \left(1+\frac{x}{2}\right)^2, \left(1+\frac{x}{3}\right)^3, \ldots, \left(1+\frac{x}{n}\right)^n, \ldots$$

has the limit e^x, or that the series

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+(-1)^{n-1}\frac{x^n}{n}+\ldots$$

has the sum $\log(1+x)$, we considered x a constant number. The functional nature of the elements of a sequence and its limit or of the terms of a series and its sum has not been taken into consideration by us at all; now we shall turn our attention to this.

We presuppose that a sequence is given, whose elements are the functions

$$f_1(x), f_2(x), f_3(x), \ldots, f_n(x), \ldots \tag{1}$$

in precisely one variable x, defined in some domain of variation for x, $\mathfrak{X} = \{x\}$†. For every x in $\mathfrak{X}$ let this sequence have a finite limit;

† Usually this will be an interval; but in the meantime we preserve the greatest generality by regarding $\mathfrak{X}$ as any infinite set of numbers.

because it is completely defined by x, it also is a function of x in $\mathfrak{X}$:

$$f(x) = \lim_{n \to \infty} f_n(x), \tag{2}$$

which we will call the limit function for the sequence (1) or for the function $f_n(x)$.

Now we shall be interested not only in the mere existence of a limit at the separate points x, but also in the functional properties of the limit function—the question of its continuity, of its derivative and integral.

We have already seen [Sec. 234] that the study of a number series and its sum is only another form of the study of a sequence of numbers and its limit. Now we consider a series, whose terms are functions in precisely one variable x in some domain:

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots . \tag{3}$$

Let this series converge for every value of x in $\mathfrak{X}$; then its sum is also some function of x: $f(x)$. This sum is defined by a limit equation of the form (2), if by $f_n(x)$ we understand the partial sum

$$f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x). \tag{4}$$

Conversely, the question of a limit function for an arbitrary given sequence (1) can be considered by examining the summation of the series (3), if we set

$$u_1(x) = f_1(x),\ u_2(x) = f_2(x) - f_1(x), \ldots,$$
$$u_n(x) = f_n(x) - f_{n-1}(x), \ldots .$$

Usually we will have to deal with functional series, because this form of considering the limit function is usually more convenient in practice.

It is also necessary to emphasize here that the object of our closest consideration will not merely be the question of the convergence of the series (3), but also the functional properties of its sum, which we mentioned above.

As it turns out, the functional properties of the limit function (the sum of the series) $f(x)$ essentially depend not only on the functional properties of the functions $f_n(x)$ (or the terms of the series),

but also on the character of the approximation of $f_n(x)$ to $f(x)$. First of all we shall concern ourselves with the study of typical possibilities, which are presented here.

264. Uniform and non-uniform convergence. Let us suppose that for all x in $\mathfrak{X}$, equation (2) holds. With the same definition of a limit this implies that: if we first fix the value of x in $\mathfrak{X}$ (so that the discussion is about a definite sequence of numbers), for any given $\varepsilon > 0$ it is possible to find a number N such that for all $n > N$ the inequality

$$|f_n(x) - f(x)| < \varepsilon \tag{5}$$

is satisfied, where by x we mean precisely that value which was fixed beforehand.

If we take another value of x, then we obtain another sequence of numbers, and—for the same ε—the number N which we have found may turn out to be unsuitable; then it would be necessary to replace it by a larger number. But x takes an infinite number of values, so that we have an infinite number of different sequences of numbers, converging to a limit. For each of these separately we find its own N; the question arises: does there exist a number N, which (for a given ε) might serve at the same time for all these sequences?

We will show by means of examples that in some cases such a number does exist, and in others it does not.

(1) First let

$$f_n(x) = \frac{x}{1+n^2x^2}, \quad f(x) = \lim_{n\to\infty} f_n(x) = 0 \quad (0 \leqslant x \leqslant 1).$$

Because here

$$0 \leqslant f_n(x) = \frac{1}{2n}\,\frac{2nx}{1+n^2x^2} \leqslant \frac{1}{2n},$$

then it is immediately clear that, for the realization of the inequality $f_n(x) < \varepsilon$, it is sufficient, whatever the value of x, to take $n > 1/2\varepsilon$. Thus, for instance, the number $N = E(1/2\varepsilon)$† in this case is suitable for all x at the same time.

(2) Now we set:

$$f_n(x) = \frac{nx}{1+n^2x^2}, \quad f(x) = \lim_{n\to\infty} f_n(x) = 0 \quad (0 \leqslant x \leqslant 1).$$

For any fixed $x > 0$ it is sufficient to take $n > E(1/x\varepsilon)$ in order that: $f_n(x) < 1/(nx) < \varepsilon$. But, on the other hand, however large we take n, for the function

† For the meaning of the symbol $E(x)$ see Vol. I, p. 30.

$f_n(x)$ in the interval [0, 1] we can always find a point, namely the point $x = 1/n$, at which its value is equal to $1/2$: $f_n(1/n) = 1/2$. Thus, on account of the increase in n it is quite impossible to make $f_n(x) < 1/2$ for all values of x from 0 to 1 at the same time. In other words, for $\varepsilon = 1/2$ there does not exist a number N which serves for all x at the same time.

In Fig. 2 are shown the graphs of these functions corresponding to $n = 4$ and $n = 40$: characteristic is the turning point of ordinate 1/2, which moves from the right to the left as n increases. Although at every vertical, taken separately, the points of the successive curves approximate to the axis of x as n increases infinitely, yet not one of these curves, on the whole, tends to this axis for the complete interval [0, 1].

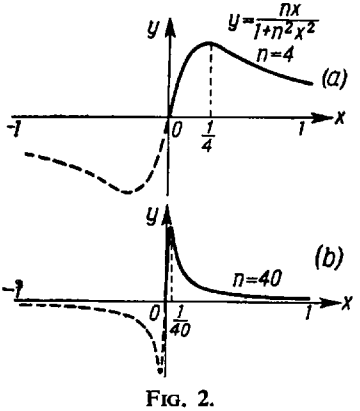

FIG. 2.

The situation with regard to the first set of functions we considered is quite different; we do not give their graphs, because when $n = 4$ or $n = 40$, for instance, they can be obtained from the graphs represented in Fig. 2 by decreasing all the ordinates, respectively, 4 or 40 times. In this case the curves at once tend to the axis or x along the whole interval [0, 1].

Now we give a fundamental definition:

If: (1) the sequence (1) *has a limit function* $f(x)$ *in* $\mathfrak{X}$ *and (2) for every number* $\varepsilon > 0$ *there exists a number* N, *not depending on* x, *such that for* $n > N$ *the inequality* (5) *is satisfied at the same time for all* x *in* $\mathfrak{X}$, *then we say that the sequence* (1) *converges* [*or the function* $f_n(x)$ *tends*] *to the function* $f(x)$ *uniformly with respect to* x *in the domain* $\mathfrak{X}$.

Thus, in the first of the examples given the function $f_n(x)$ tends to zero uniformly with respect to x in the interval [0, 1], and in the second it does not.

We will consider some further examples of non-uniform convergence.

(3) If

$$f_n(x) = x^n \qquad (0 \leqslant x \leqslant 1),$$

then

$$f(x) = \lim_{n \to \infty} f_n(x) = 0$$

where $x < 1$ and $f(1) = 1$. In this case the impossibility of the inequality $x^n < \varepsilon$ (where $\varepsilon < 1$) holding simultaneously for all $x < 1$ is evident, if only from the fact that $x^n \to 1$, if (for fixed n) $x \to 1$. Figure 3 gives a representation of the peculiar way in which the inequality is violated: here the limit function changes by jumps, but the highest point is fixed.

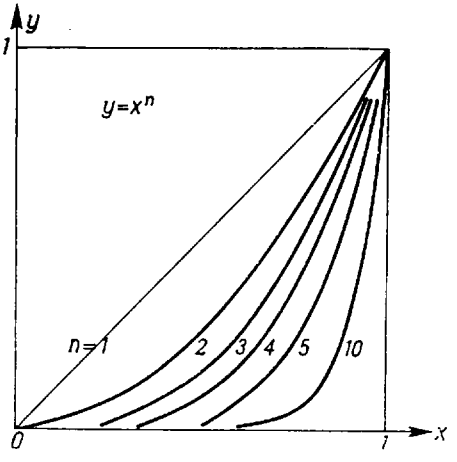

FIG. 3.

(4) Finally, let

$$f_n(x) = 2n^2x \cdot e^{-n^2x^2} \qquad (0 \leqslant x \leqslant 1).$$

The impossibility of obtaining a uniform approximation in $[0, 1]$ to the limit function, which is here identically equal to 0, follows merely from the fact that, for any n,

$$f_n\left(\frac{1}{n}\right) = \frac{2n}{e}.$$

This time the height of the highest points, which interfere with the uniform tendency to zero, in fact increases to infinity with n.

Nowweextend all that has been said above about the convergence of functions to the case of a functional series (3).

Presupposing that the series converges, we take into consideration its sum $f(x)$, the partial sum $f_n(x)$ [see (4)] and, finally, its remainder after the nth term

$$\varphi_n(x) = \sum_{k=n+1}^{\infty} u_k(x) = f(x) - f_n(x).$$

For any fixed x

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{and} \quad \lim_{n\to\infty} \varphi_n(x) = 0.$$

If the partial sum $f_n(x)$ tends to the sum of the series $f(x)$ uniformly with respect to x in the domain $\mathfrak{X}$ [or, which is the same thing, the remainder series $\varphi_n(x)$ tends uniformly to zero], then we say that the series (3) *converges uniformly in this domain.*

This definition is clearly equivalent to the following:

The series (3), *which converges for all x in the domain $\mathfrak{X}$, is said to converge uniformly in this domain, if for every number $\varepsilon > 0$ there exists a number N not depending on x such that when $n > N$ the inequality*

$$|f_n(x) - f(x)| < \varepsilon \quad \text{or} \quad |\varphi_n(x)| < \varepsilon \tag{6}$$

is satisfied at the same time for all x in $\mathfrak{X}$.

Examples of uniformly and non-uniformly convergent series can be constructed by altering the examples of sequences given above. We add to them the following simple example.

(5) We consider the progression $\sum_{n=1}^{\infty} x^{n-1}$; it converges in the open interval $\mathfrak{X} = (1, -1)$. For any x in $\mathfrak{X}$ the remainder after the nth term has the form:

$$\varphi_n(x) = \frac{x^n}{1-x}.$$

If n is fixed arbitrarily, then clearly:

$$\lim_{x\to -1+0} |\varphi_n(x)| = \frac{1}{2}, \quad \lim_{x\to 1-0} \varphi_n(x) = \infty.$$

Then, as the second of these shows, to realize the inequality

$$|\varphi_n(x)| < \varepsilon \quad (\text{if } \varepsilon < 1/2)$$

for all x at the same time, with the same number n, is impossible. The convergence of the progression in the interval $(-1, 1)$ is non-uniform; but this does occur in the intervals $(-1, 0]$ and $[0, 1)$ separately.

265. The condition for uniform convergence. The Bolzano–Cauchy theorem [Sec. 52], which establishes a condition for the existence of a finite limit for a given number sequence ("the principle of convergence"), naturally leads to the following condition for the uniform convergence of the number sequence (1) given in the domain $\mathfrak{X}$:

In order that the sequence (1)*: (1) should have a limit function and (2) should converge to this function uniformly with respect to x in the domain $\mathfrak{X}$, it is necessary and sufficient that for every number $\varepsilon > 0$ there should exist a number N not depending on x such that for $n > N$ and any $m = 1, 2, 3, \ldots$ the inequality*

$$|f_{n+m}(x) - f_n(x)| < \varepsilon \tag{7}$$

holds for all x in $\mathfrak{X}$ at the same time.

This requirement can be briefly formulated thus: the principle of convergence for the sequence (1) must be realized "uniformly" for all x in $\mathfrak{X}$.

Proof. Necessity. If the sequence (1) has a limit function $f(x)$ and converges to it uniformly in $\mathfrak{X}$, then for a given $\varepsilon > 0$ we can find a number N not depending on x such that when $n > N$ it will be true that

$$|f_n(x) - f(x)| < \tfrac{1}{2}\varepsilon$$

for all x. Analogously also

$$|f_{n+m}(x) - f(x)| < \tfrac{1}{2}\varepsilon \qquad (m = 1, 2, 3, \ldots),$$

and from both these inequalities follows (7).

Sufficiency. Let the condition given in the theorem be satisfied. Then, whatever value of x in $\mathfrak{X}$ is taken, in the place of the sequence (1) we will have a number sequence, for which the Bolzano–Cauchy condition is satisfied. Hence, for this sequence there exists a finite limit, from which is proved the existence of a limit function $f(x)$ for the sequence (1).

Now, taking an arbitrary $n > N$ and x in $\mathfrak{X}$, we increase m without limit in the inequality (7) (for constant n and x). Taking limits, we obtain:

$$|f(x) - f_n(x)| \leqslant \varepsilon.$$

This establishes that $f_n(x)$ tends uniformly to $f(x)$.

It is not difficult to rephrase the condition we have proved for the case of a functional series:

In order that the series (3) *should converge uniformly in the domain* $\mathcal{X}$, *it is necessary and sufficient that for every number* $\varepsilon > 0$ *there should exist a number* N *not depending on* x *such that for* $n > N$ *and any* $m = 1, 2, 3, \ldots$ *the inequality*

$$\left| \sum_{k=n+1}^{n+m} u_k(x) \right| = |u_{n+1}(x) + u_{n+2}(x) + \ldots + u_{n+m}(x)| < \varepsilon \qquad (8)$$

holds for all x *in* $\mathcal{X}$ *at the same time.*

In order to establish in practice the uniform convergence of actual sequences or series, we use sufficiency tests more convenient to apply, which are usually formulated for series.

Here is the simplest test, which is also the one most often used:

WEIERSTRASS'S TEST. *If the terms of the functional series* (3) *satisfy in the domain* $\mathcal{X}$ *the inequality*

$$|u_n(x)| \leqslant c_n \quad (n = 1, 2, 3, \ldots), \qquad (9)$$

where the c_n *are the terms of some convergent number series*

$$\sum_{n=1}^{\infty} c_n = c_1 + c_2 + \ldots + c_n + \ldots, \qquad (C)$$

then the series (3) *converges uniformly in* $\mathcal{X}$.

When the inequality (9) occurs, we say that the series (3) is "majorized" by the series (C), or that the series (C) serves as a "major" series for (3).

In fact, from (9) we obtain the inequality

$$|u_{n+1}(x) + u_{n+2}(x) + \ldots + u_{n+m}(x)| \leqslant c_{n+1} + c_{n+2} + \ldots + c_{n+m},$$

valid for all x in the domain $\mathcal{X}$ at the same time. By the principle of convergence, which we apply to the number series (C), for any $\varepsilon > 0$ we can find an N such that for $n > N$ the right-hand side of the preceding inequality will be less than ε, and hence so will the left-hand side, moreover for all x at once. Hence, by virtue of the condition proved above, our assertion follows.

Thus, for instance, in any interval the series

$$\sum_{n=1}^{\infty} a_n \sin nx, \qquad \sum_{n=1}^{\infty} a_n \cos nx$$

converge uniformly, if the series $\sum_{n=1}^{\infty} a_n$ merely converges absolutely. In fact,

$$|a_n \sin nx| \leqslant |a_n|, \quad |a_n \cos nx| \leqslant |a_n|,$$

so that the part of the "major series" is played here by the series $\sum_{n=1}^{\infty} |a_n|$.

§ 2. The functional properties of the sum of a series

266. The continuity of the sum of a series. Now we turn to the study of the functional properties of the sum of a series consisting of functions, in connection with the properties of the latter. We have already remarked above on the equivalence between considering sequences and considering infinite series. In our exposition we prefer the latter point of view, because in applications one meets infinite series almost exclusively. Carrying over to the case of sequences of functions what has been said about functional series presents no difficulties.

The concept of uniform convergence given above will play a decisive role in all that follows, so that its importance must be stressed.

We will begin with the question of the continuity of the sum of a series consisting of continuous functions. The reader knows that the sum of a finite number of continuous functions is continuous [Sec. 62]; is a similar assertion also true for the case of an infinite number of summands? The following simple example shows that this is not always so.

We consider the series

$$\sum_{n=1}^{\infty} x^{n-1}(1-x) = (1-x) + x(1-x) + \ldots + x^{n-1}(1-x) + \ldots$$

$$(0 \leqslant x \leqslant 1).$$

When $x = 1$, all the terms of the series, and with them the sum of the series, become zero; when $x < 1$, summing the progression, we obtain 1. Although the terms of the series are continuous in the interval [0, 1], yet the sum of the series at the point $x = 1$ has a discontinuity! We note that the convergence of the series here is not uniform, because the remainder of the series after the nth term, which is equal to x^n (for $x < 1$), tends non-uniformly to 0 [Sec. 264, *(3)*].

We have

THEOREM 1. *If the functions* $u_n(x)$ $(n = 1, 2, 3, \ldots)$ *are defined and continuous in the interval* $\mathfrak{X} = [a, b]$, *and the series*

$$\sum_{n=1}^{\infty} u_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots \tag{1}$$

converges uniformly in $\mathfrak{X}$ *to the sum* $f(x)$, *then this sum will also be continuous in the interval* $\mathfrak{X}$.

Proof. We take any point x_0 in the interval $\mathfrak{X}$ and establish the continuity of the function $f(x)$ at this point. By our previous notation, we have for any $n = 1, 2, 3, \ldots$ and any x in $\mathfrak{X}$:

$$f(x) = f_n(x) + \varphi_n(x) \tag{2}$$

and, in particular,

$$f(x_0) = f_n(x_0) + \varphi_n(x_0),$$

hence

$$|f(x) - f(x_0)| \leqslant |f_n(x) - f_n(x_0)| + |\varphi_n(x)| + |\varphi_n(x_0)|. \tag{3}$$

Now we take an arbitrary $\varepsilon > 0$. In view of the uniform convergence of the series we can fix a number n so that the inequality

$$|\varphi_n(x)| < \frac{\varepsilon}{3} \tag{4}$$

is satisfied for all values of x in the interval $\mathfrak{X}$ (and amongst them for $x = x_0$). We note that for a fixed n the function $f_n(x)$ is the sum of a definite finite number of functions $u_k(x)$, continuous at the point $x = x_0$. Therefore it is also continuous at this point, and for the given $\varepsilon > 0$ we can find a $\delta > 0$ such that when $|x - x_0| < \delta$ it will be true that

$$|f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}. \tag{5}$$

Then, in view of (3), (4) and (5), the inequality $|x - x_0| < \delta$ implies

$$|f(x) - f(x_0)| < \varepsilon,$$

which proves the theorem†.

† In fact, it is proved here that from the continuity of the terms of the series at a definite point follows the continuity of its sum at the same point.

Remark. We saw in an example that the assertion of the theorem can be shown to be untrue if one omits the requirement that the series converge uniformly. However, the uniform continuity appears in the theorem only as a sufficiency condition, and it is not necessary to suppose that this condition is necessary for the continuity of the sum of the series. For instance, the series

$$\left.\begin{aligned} &\sum_{n=1}^{\infty} 2x[n^2e^{-n^2x^2} - (n-1)^2e^{-(n-1)^2x^2}], \\ &\sum_{n=1}^{\infty}\left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}\right] \end{aligned}\right\} \tag{6}$$

[cf. Sec. 264, *(4)* and *(2)*] have a continuous sum in the interval [0, 1] identically equal to zero, although in the given interval they both converge non-uniformly.

It is easy to rephrase the theorem we have proved for the case of a sequence of functions:

THEOREM 1*. *If the sequence of functions*

$$f_1(x), f_2(x), \ldots, f_n(x), \ldots, \tag{7}$$

defined and continuous in the interval $\mathcal{X} = [a, b]$, *converges to the limit function* $f(x)$ *uniformly in* $\mathcal{X}$, *then* $f(x)$ *is also continuous in* $\mathcal{X}$.

267. The case of positive series. For series of this special type, as Dini† proved, uniform convergence can be shown to be not only a sufficient, but also a necessary, condition for the continuity of the sum of the series:

THEOREM 2. *Let the terms of the series* (1) *be continuous in the interval* $\mathcal{X} = [a, b]$ *and non-negative. If the series has the sum* $f(x)$, *also continuous in the interval* $\mathcal{X}$, *then the series converges uniformly in this interval.*

Proof. We consider the remainder of the series (1):

$$\varphi_n(x) = \sum_{k=n+1}^{\infty} u_k(x) = f(x) - f_n(x).$$

The function $\varphi_n(x)$, being the difference of two continuous functions, is also continuous. Because the terms of the series are positive, the sequence $\{\varphi_n(x)\}$, for constant x, is decreasing (non-increasing):

$$\varphi_1(x) \geqslant \varphi_2(x) \geqslant \ldots \geqslant \varphi_n(x) \geqslant \varphi_{n+1}(x) \geqslant \ldots.$$

† Uliss Dini (1845–1918) was an Italian mathematician.

Finally, in so far as the series (1) converges in the interval $\mathcal{X}$, for any constant x

$$\lim_{n\to\infty} \varphi_n(x) = 0.$$

In order to establish the uniform convergence of the series, it is sufficient to prove that for every $\varepsilon > 0$ there exists at least one value of n for which $\varphi_n(x) < \varepsilon$ at the same time for all x (because then for greater values of n this inequality will certainly be satisfied).

We will prove this by *reductio ad absurdum*. We presuppose that for some $\varepsilon > 0$ such a number n does not exist. Then for any $n = 1, 2, 3, \ldots$ we can find in the interval $\mathcal{X}$ a value $x = x_n$ such that $\varphi_n(x_n) \geqslant \varepsilon$. To the sequence $\{x_n\}$, all of whose elements are contained in the finite interval $\mathcal{X}$, we apply the Bolzano–Weierstrass lemma [Sec. 51] and we single out from it the partial sequence $\{x_{n_k}\}$, converging to the limit x_0.

By virtue of the continuity of $\varphi_m(x)$ we have:

$$\lim_{k\to\infty} \varphi_m(x_{n_k}) = \varphi_m(x_0),$$

whatever the value of m. On the other hand, for any m, for sufficiently large k:

$$n_k \geqslant m, \quad \text{so that} \quad \varphi_m(x_{n_k}) \geqslant \varphi_{n_k}(x_{n_k}) \geqslant \varepsilon.$$

Taking limits here as $k \to \infty$, we find that

$$\lim_{k\to\infty} \varphi_m(x_{n_k}) = \varphi_m(x_0) \geqslant \varepsilon.$$

But this inequality, which holds for any m, contradicts the fact that

$$\lim_{m\to\infty} \varphi_m(x_0) = 0.$$

The theorem is proved.

If we rephrase Dini's theorem for the case of a sequence, then we obtain:

THEOREM 2*. *Let the sequence* (7) *of functions continuous in the interval* $\mathcal{X} = [a, b]$ *tend, as* $n \to \infty$, *to the limit function* $f(x)$, *which is monotonically increasing, so that*

$$f_{n+1}(x) \geqslant f_n(x).$$

If the function $f(x)$ *is also continuous in* $\mathcal{X}$, *then* $f_n(x)$ *converges uniformly to* $f(x)$ *in* $\mathcal{X}$.

268. Termwise transition to a limit. We have already introduced one theorem which is a generalization of Theorem 1. In it $\mathfrak{X}$ is an arbitrary infinite set, which has an accumulation point a (finite or not) [Sec. 32]; this point need not belong to the set $\mathfrak{X}$.

THEOREM 3. *Let each of the functions* $u_n(x)$ $(n = 1, 2, 3, \ldots)$ *be defined in the domain* $\mathfrak{X}$ *and have, as* x *tends to* a, *a finite limit:*

$$\lim_{x \to \infty} u_n(x) = c_n. \tag{8}$$

If the series (1) *converges uniformly in the domain* $\mathfrak{X}$, *then (1) the series formed from these limits*:

$$\sum_{n=1}^{\infty} c_n = C \tag{C}$$

converges, and (2) the sum of the series (1), $f(x)$, *also has a limit as* $x \to a$, *namely*:

$$\lim_{x \to \infty} f(x) = C. \tag{9}$$

Proof. By the condition for uniform convergence in Sec. 265, for $\varepsilon > 0$ chosen arbitrarily there exists a number N such that for $n > N$ and $m = 1, 2, 3, \ldots$ the inequality (8) on p. 72 is satisfied for all x in $\mathfrak{X}$. Taking limits here as $x \to a$, using (8), we find that

$$|c_{n+1} + c_{n+2} + \ldots + c_{n+m}| \leqslant \varepsilon,$$

so that the series (C) satisfies the condition for convergence in Sec. 242.

If C, C_n and γ_n denote, as usual, its sum, partial sum and remainder, then

$$C = C_n + \gamma_n.$$

Subtracting this equation term by term from (2), it is easy to show that

$$|f(x) - C| \leqslant |f_n(x) - C_n| + |\varphi_n(x)| + |\gamma_n|. \tag{10}$$

In view of the uniform convergence of the series (1) and the convergence of the series (C), for any $\varepsilon > 0$ one can fix n large enough such that, for all x in $\mathfrak{X}$,

$$|\varphi_n(x)| < \varepsilon/3, \quad \text{and also} \quad |\gamma_n| < \varepsilon/3. \tag{11}$$

Because, clearly,

$$\lim_{x\to a} f_n(x) = \lim_{x\to a} \sum_{k=1}^{n} u_k(x) = \sum_{k=1}^{n} c_k = C_n,$$

then—if we limit ourselves to the case of finite a—we can find a $\delta > 0$ such that when $|x-a| < \delta$ it will be true that:

$$|f_n(x) - C_n| < \varepsilon/3. \tag{12}$$

Then, for the values of x indicated, by virtue of (10), (11) and (12), the inequality

$$|f(x) - C| < \varepsilon$$

will be satisfied, which leads to (9)†.

The equation (9) can be written in the form [see (8)]:

$$\lim_{x\to a} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \{\lim_{x\to a} u_n(x)\};$$

thus, when there is uniform convergence, *the limit of the sum of a functional series is equal to the sum of the series composed of the limits of its terms*, or, in other words, *we can suppose that in a functional series the transition to the limit is "termwise"*.

Example. As an example of the application of this general theorem we present a derivation of the logarithmic series [Sec. 256, (21)] known to the reader, starting from the binomial series [Sec. 258, (26)], with the help of the limit relation [Sec. 65, *(2)*],

$$\log a = \lim_{k\to\infty} k(\sqrt[k]{a} - 1).$$

(Essentially, this is just how Euler approaches the logarithmic series in his *Introduction to Analysis*, but, of course, without rigorously laying down the foundations!)

We set $a = 1 + x$ (where $|x| < 1$) and instead of $(1+x)^{1/k}$ we substitute this expansion

$$(1+x)^{\frac{1}{k}} = 1 + \frac{1}{k}x + \frac{\frac{1}{k}\left(\frac{1}{k}-1\right)}{1\cdot 2}x^2 + \ldots + \frac{\frac{1}{k}\left(\frac{1}{k}-1\right)\ldots\left(\frac{1}{k}-n+1\right)}{1\cdot 2\cdot\ldots\cdot n}x^n + \ldots.$$

† The reader will recognize that this is the same argument as that applied n the proof of Theorem 1.

Then $\log(1+x)$ is represented as the limit, as $k\to\infty$, of the expression

$$k[(1+x)^{\frac{1}{k}}-1]=x-\frac{x^2}{2}\left(1-\frac{1}{k}\right)+\frac{x^3}{3}\left(1-\frac{1}{k}\right)\left(1-\frac{1}{2k}\right)-\ldots$$
$$+(-1)^{n-1}\frac{x^n}{n}\left(1-\frac{1}{k}\right)\left(1-\frac{1}{2k}\right)\ldots\left(1-\frac{1}{(n-1)k}\right)+\ldots. \quad (13)$$

We emphasize that here x denotes a constant number. The terms of this series contain, as a variable, the natural parameter k†. The series (13) converges uniformly in the whole domain of variation of k; this follows (by Weierstrass's test) from the fact that it is majorized by the series

$$|x|+\frac{|x|^2}{2}+\frac{|x|^3}{3}+\ldots+\frac{|x|^n}{n}+\ldots \quad (x=\text{const},\ |x|<1),$$

which is independent of k. Thus, by Theorem 3, one can let $k\to\infty$ termwise in the series (13), which then reduces to the logarithmic series.

269. Termwise integration of series. Now we turn to the consideration of the question of integrating the sum of a convergent functional series.

THEOREM 4. *If the functions $u_n(x)$ $(n=1,2,3,\ldots)$ are continuous in the interval $\mathfrak{X}=[a,b]$, and the series* (1) *formed from them converges uniformly in this interval, then the integral of the sum $f(x)$ can be represented in the following way:*

$$\int_a^b f(x)\,dx=\sum_{n=1}^{\infty}\int_a^b u_n(x)\,dx$$
$$=\int_a^b u_1(x)\,dx+\int_a^b u_2(x)\,dx+\ldots+\int_a^b u_n(x)\,dx+\ldots. \quad (14)$$

Proof. The existence of all these integrals follows from the continuity of the functions $u_n(x)$ and $f(x)$ [Sec. 266, Theorem 1]. Integrating the identity

$$f(x)=u_1(x)+u_2(x)+\ldots+u_n(x)+\varphi_n(x)$$

in the interval $[a,b]$, we obtain:

$$\int_a^b f(x)\,dx=\int_a^b u_1(x)\,dx+\int_a^b u_2(x)\,dx+\ldots+\int_a^b u_n(x)\,dx+\int_a^b \varphi_n(x)\,dx.$$

† We recall that the domain of variation $\mathfrak{X}$ of the variable x, which was discussed in Theorem 3, could be any one desired; in particular, it could be reduced to the set of natural numbers so that $a=+\infty$.

Thus, the sum of n terms of the series (14) differs from the integral $\int_a^b f(x)dx$ by the remainder $\int_a^b \varphi_n(x)dx$. In order to prove the expansion (14) we need only establish that

$$\lim_{n\to\infty} \int_a^b \varphi_n(x)dx = 0. \tag{15}$$

By virtue of the uniform convergence of the series (1), for any $\varepsilon > 0$ we can find a number N such that, when $n > N$,

$$|\varphi_n(x)| < \varepsilon$$

for all x in the interval considered. Then for the same values of n it will be true that:

$$\left|\int_a^b \varphi_n(x)dx\right| \leqslant \int_a^b |\varphi_n(x)|\,dx < (b-a)\varepsilon,$$

which proves the limit relation (15).

Equation (14) can be rewritten in the form

$$\int_a^b \left\{\sum_{n=1}^{\infty} u_n(x)\right\} dx = \sum_{n=1}^{\infty} \left\{\int_a^b u_n(x)\,dx\right\},$$

so that, in the case of a uniformly convergent functional series, *the integral of the sum of the series is equal to the sum of the series formed from the integrals of its sums*, or, in other words, *we can admit termwise integration of the series.*

Remark. As in the case of Theorem 1, the requirement of uniform convergence is essential for the validity of the expansion (14), i.e. it cannot simply be omitted, but nevertheless it is not necessary. The series (6), considered in Sec. 266, will illustrate this point. Both series converge non-uniformly in the interval [0, 1] to the function $f(x)$. But integrating the first termwise, we obtain as the sum of the series of integrals

$$\lim_{n\to\infty} \int_0^1 2n^2x \cdot e^{-n^2x^2}dx = \lim_{n\to\infty} (1 - e^{-n^2}) = 1,$$

although

$$\int_0^1 f(x)dx = 0;$$

for the second series we find analogously

$$\lim_{n\to\infty}\int_0^1 \frac{nx}{1+n^2x^2}dx = \lim_{n\to\infty}\frac{\log(1+n^2)}{2n} = 0 = \int_0^1 f(x)dx.$$

Example. We consider the question of expanding the so-called completely elementary integral of the first kind

$$F(k) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{(1-k^2\sin^2\varphi)}}$$

by powers of the modulus $k\,(0<k<1)$.

Starting from the expansion of $(1+x)^{1/2}$ in a binomial series [Sec. 258] and letting $x = -k^2\sin^2\varphi$ in it, we obtain

$$\frac{1}{\sqrt{(1-k^2\sin^2\varphi)}} = 1 + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{2n!!}k^{2n}\sin^{2n}\varphi.$$

This series converges uniformly with respect to φ, because it is majorized by the convergent progression $1+\sum_{n=1}^{\infty}k^{2n}$; hence, termwise integration with respect to φ in the interval $[0,\pi/2]$ is permissible. Carrying out this integration and making use of the well-known integral [Sec. 187]

$$\int_0^{\pi/2}\sin^{2n}\varphi\,d\varphi = \frac{(2n-1)!!}{2n!!}\cdot\frac{\pi}{2},$$

we obtain

$$F(k) = \frac{\pi}{2}\left\{1+\sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{2n!!}\right]^2\cdot k^{2n}\right\}.$$

This expansion—particularly for small k—can be used for approximate calculations.

We quote an analogue to Theorem 4 for sequences:

THEOREM 4*. *If the sequence* (7) *of functions continuous in the interval* $\mathfrak{X} = [a,b]$ *converges uniformly to the limit function* $f(x)$ *in* $\mathfrak{X}$, *then*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\int_a^b f_n(x)\,dx. \tag{16}$$

This equation can be rewritten in the form

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b \{\lim_{n\to\infty} f_n(x)\}\,dx,$$

so that it turns out that the limit of the integral may immediately be related to the function under the integral. In this case we say that *it is permissible to take limits under the integral sign.*

In Chapter 18 we shall return to this question and discuss it in more general terms.

270. Termwise differentiation of series. With the help of Theorem 4 in the preceding section, it is easy to prove the following

THEOREM 5. *Let the functions* $u_n(x)$ $(n = 1, 2, 3, \ldots)$ *be defined in the interval* $\mathcal{X} = [a, b]$ *and have continuous derivatives* $u_n'(x)$ *in it. If over this interval the series* (1) *converges, and also the series composed of its derivatives*

$$\sum_{n=1}^{\infty} u_n'(x) = u_1'(x) + u_2'(x) + \ldots + u_n'(x) + \ldots \tag{17}$$

converges uniformly, then the sum $f(x)$ *of the series* (1) *also has a derivative in* $\mathcal{X}$, *where*

$$f'(x) = \sum_{n=1}^{\infty} u_n'(x). \tag{18}$$

Proof. We denote by $f^*(x)$ the sum of series (17); by virtue of Theorem 1, this will be a continuous function of x. Now using Theorem 4, we integrate the series (17) termwise in the interval from a to an arbitrary value of x in $\mathcal{X}$; we obtain

$$\int_a^x f^*(t)\,dt = \sum_{n=1}^{\infty} \int_a^x u_n'(t)\,dt.$$

But, clearly, $\int_a^x u_n'(t)dt = u_n(x) - u_n(a)$, so that

$$\int_a^x f^*(t)\,dt = \sum_{n=1}^{\infty} [u_n(x) - u_n(a)]$$

$$= \sum_{n=1}^{\infty} u_n(x) - \sum_{n=1}^{\infty} u_n(a) = f(x) - f(a).$$

From the continuity of the integrand the integral on the right has a derivative equal to $f^*(x)$ [Sec. 183, *(12)*], so the function $f(x)$, which differs from the integral by a constant, has the same derivative.

Equation (18) can be rewritten (if, following Cauchy, we use D to denote the derivative) in the form

$$D\left\{\sum_{n=1}^{\infty} u_n(x)\right\} = \sum_{n=1}^{\infty} Du_n(x).$$

Thus, under the conditions indicated, *the derivative of the sum of the series is equal to the sum of the series consisting of the derivatives of its terms,* or, in other words, "*termwise*" *differentiation of the series is permissible.*

Remark. We consider the series

$$\sum_{n=1}^{\infty} [e^{-(n-1)^2x^2} - e^{-n^2x^2}]$$

and

$$\frac{1}{2}\log(1+x^2) + \sum_{n=2}^{\infty}\left[\frac{1}{2n}\log(1+n^2x^2) - \frac{1}{2(n-1)}\log(1+(n-1)^2x^2)\right].$$

The first of these converges to 0 when $x = 0$ and to 1 at the remaining points, but the sum of the second is everywhere equal to 0. If we differentiate them termwise, then we obtain the series (6) [Sec. 266] already known to us, which converge to 0 in the whole interval $[0, 1]$, but both of them non-uniformly. In the first case the series of the derivatives also converges when $x = 0$, where the sum of the primitive series cannot have a derivative, because there is a discontinuity at this point. In the second case, on the contrary, termwise differentiation gives the correct result everywhere. These examples illustrate the significance of the requirement that the series of derivatives should converge uniformly: it is essential, but not necessary.

We leave to the reader the rephrasing of Theorem 5 for the case of a functional sequence.

All these theorems on termwise passage to a limit, termwise integration and differentiation establish an analogy between functional series and the sums of a finite number of functions. This analogy, however, is limited by well-known conditions, in the characterizing of which uniform convergence occupies an intrinsic position.

271. An example of a continuous function without a derivative. To conclude this section we will mention, using a functional series, an example of a continuous function, which at no point has a finite derivative.

The first example of this type was discovered by Weierstrass, who published it in 1875†; his function is defined by the series:

$$f(x) = \sum_{n=0}^{\infty} a^n \cos b^n \pi x,$$

where $0 < a < 1$, and b is an odd natural number (where $ab > 1 + 3\pi/2$). This series is majorized by the convergent progression $\sum_0^\infty a^n$, hence [Sec. 265; Sec. 266, Theorem 1], it converges uniformly, and its sum is everywhere a continuous function of x. By painstaking examination Weierstrass succeeded in showing that nevertheless there does not exist a finite derivative for it at a single point.

We will quote a more simple example due to van der Waerden, constructed essentially from the same idea: only the oscillating curves $y = \cos \omega x$ are replaced by oscillating breaks.

Thus, we denote by $u_0(x)$ the absolute value of the difference between the number x and the whole number nearest to it. This function will be linear in every interval of the form $[s/2, (s+1)/2]$, where s is a whole number; it is continuous and has period 1. Its graph is broken; it is shown in Fig. 4*a*; the separate broken links have slope ± 1.

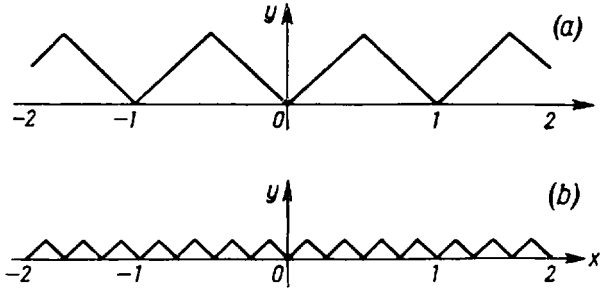

FIG. 4.

Next we set, for $k = 1, 2, 3, \ldots$,

$$u_k(x) = u_0(4^k x)/4^k.$$

This function will be linear in intervals of the form $[s/(2 \cdot 4^k), (s+1)/(2 \cdot 4^k)]$; it is also continuous and has period $1/4^k$. Its graph is also broken, but with smaller "teeth"; in Fig. 4*b*, for instance, is shown the graph of the function $u_1(x)$. In all cases the slopes of the separate broken links are also equal to ± 1 here.

† However, here he was following Bolzano, who constructed a similar function earlier (1830).

Now we will define, for all real values of x, a function $f(x)$ by means of the equation

$$f(x) = \sum_{k=0}^{\infty} u_k(x).$$

Since $0 \leqslant u_k(x) \leqslant 1/(2 \cdot 4^k)$ $(k = 0, 1, 2, \ldots)$, the series is majorized by the convergent progression $\sum_{0}^{\infty} [1/(2 \cdot 4^k)]$, so (as in the case of Weierstrass's function) the series converges uniformly, and the function $f(x)$ is continuous everywhere.

We shall consider any value $x = x_0$. Calculating it to within $1/(2 \cdot 4^n)$ (where $n = 0, 1, 2, \ldots$) above or below its value, we can include it between numbers of the form:

$$\frac{s_n}{2 \cdot 4^n} \leqslant x_0 < \frac{s_n + 1}{2 \cdot 4^n},$$

where s_n is a natural number. It is evident that the closed intervals

$$\Delta_n = \left[\frac{s_n}{2 \cdot 4^n}, \frac{s_n + 1}{2 \cdot 4^n}\right] \quad (n = 0, 1, 2, \ldots)$$

are imbedded one inside the other. In each of them we can find a point x_n such that its distance from the point x_0 is equal to half the length of the interval:

$$|x_n - x_0| = \frac{1}{4^{n+1}};$$

it is clear that as n increases the variable $x_n \to x_0$.

Now we form a ratio by equating

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} = \sum_{k=0}^{\infty} \frac{u_k(x_n) - u_k(x_0)}{x_n - x_0}.$$

But, when $k > n$, the number $1/4^{n+1}$ is a whole multiple of the period $1/4^k$ of the function $u_k(x)$, so that $u_k(x_n) = u_k(x_0)$, the corresponding terms of the series tend to 0 and can be omitted. If however $k \leqslant n$, then the function $u_k(x)$, linear in the interval Δ_k, will also be linear in the interval Δ_n contained in it, where

$$\frac{u_k(x_n) - u_k(x_0)}{x_n - x_0} = \pm 1 \quad (k = 0, 1, \ldots, n).$$

Thus, we have, finally

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} = \sum_{k=0}^{n} (\pm 1);$$

in other words, this ratio is equal to an even whole number when n is odd and to an odd whole number when n is even. Hence it is clear that as $n \to \infty$ the extended ratio cannot tend to any finite limit, so that our function does not have a finite derivative when $x = x_0$.

§ 3. Power series and series of polynomials

272. The interval of convergence of a power series. The theory expounded in the preceding section has an important application in the study of the properties of power series; the latter are of the form, when arranged simply by powers of the variable x:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots, \qquad (1)$$

or—in the general case—by powers of the binomial $x - x_0$:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + \ldots + a_n (x - x_0)^n + \ldots \qquad (2)$$

($a_0, a_1, a_2, \ldots$ here denote constant coefficients). Already we have repeatedly met similar series in actual cases (see in particular § 6, Chapter 15). Now we will undertake the study of a general form, so to speak, of this same analytical process for the representation of functions. Clearly one can confine oneself to the series (1), because the series (2) reduces to (1) by a change of variable.

We will attempt first of all to clarify the structure of "the domain of convergence" of a power series, i.e. the set $\mathfrak{X} = \{\bar{x}\}$ of those values $x = \bar{x}$ of the variable, for which the series (1) converges. For this purpose let us establish the following

LEMMA. *If the series* (1) *converges for the value* $x = \bar{x}$, *distinct from* 0, *then it converges absolutely for any value of* x *satisfying the inequality:* $|x| < |\bar{x}|$.

From the convergence of the series:

$$\sum_{n=0}^{\infty} a_n \bar{x}^n = a_0 + a_1 \bar{x} + a_2 \bar{x}^2 + \ldots + a_n \bar{x}^n + \ldots$$

it follows that its general term tends to 0 [Sec. 235, *(5)*], and hence is bounded [Sec. 36, *(5)*]:

$$|a_n \bar{x}^n| \leqslant M \qquad (n = 0, 1, 2, \ldots). \qquad (3)$$

Now let us take any x for which $|x| < |\bar{x}|$, and let us form the series

$$\sum_{n=0}^{\infty} |a_n x^n| = |a_0| + |a_1 x| + |a_2 x^2| + \ldots + |a_n x^n| + \ldots. \qquad (4)$$

Because [see (3)]

$$|a_n x^n| = a_n \bar{x}^n \left|\frac{x}{\bar{x}}\right|^n \leqslant M \left|\frac{x}{\bar{x}}\right|^n,$$

the terms of the series (4) are also less than the corresponding terms of the convergent geometrical progression (with common ratio $|x/\bar{x}| < 1$):

$$M + M\left|\frac{x}{\bar{x}}\right| + M\left|\frac{x}{\bar{x}}\right|^2 + \ldots + M\left|\frac{x}{\bar{x}}\right|^n + \ldots,$$

then, by Theorem 1, Sec. 237, the series (4) converges. In such a case, as we know, the series (1) converges absolutely, thus proving the lemma.

When $x = 0$, clearly, every series (1) converges. But there exist power series which—apart from this—do not converge for any value of x. As an example of such an "everywhere divergent" series one can take the series $\sum_1^\infty n! x^n$, whose divergence one can easily verify with the help of d'Alembert's test. Such series are of no interest to us.

Now we presuppose that among the values $\bar{x}$ of the variable for which the series (1) converges, there exist some distinct from 0. We consider the set $\{|\bar{x}|\}$; it will either be bounded above or not.

In the latter case, whatever value of x we take, it is necessary to find an $\bar{x}$ such that $|x| < |\bar{x}|$, and then, by the lemma, for the value of x taken the series converges absolutely. The series is "everywhere convergent".

Now let the set $\{|\bar{x}|\}$ be bounded above, and let R be its upper bound (so that $0 < R < \infty$). If $|x| > R$, then this value of x is known to differ from all $\bar{x}$, and the series diverges. Now let us take any x for which $|x| < R$. By the definition of a bound [Sec. 6], it is necessary to find an $\bar{x}$ such that $|x| < |\bar{x}| \leqslant R$; but by the lemma this again implies the absolute convergence of the series (1).

Thus we have proved the general

THEOREM. *For every power series* (1), *provided only that it is not everywhere divergent, there exists a positive number* R *(it may also be* $+\infty$*) such that the series converges absolutely for* $|x| < R$, *and the series diverges for* $|x| > R$ *(if* $R < \infty$*).* This number R is called the *radius of convergence* of the series.

By this means we solve the problem of the "domain of convergence" $\mathfrak{X}$ of the series: it is a continuous interval from $-R$ to R; only about its end points is it impossible to make a general assertion: as we saw from the examples, there it is possible for there to be convergence (absolute or not), or divergence. The interval $\mathfrak{X}$ is called the *interval of convergence* of the series.

For an everywhere divergent series, we may take $R = 0$: its "interval of convergence" coincides with the point $x = 0$.

Examples. (1) For the series

$$1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

$R = \infty$, the interval of convergence is $(-\infty, +\infty)$ [Sec. 253].

(2) In the case of the progression

$$1 + \sum_{n=1}^{\infty} x^n$$

$R = 1$, the interval of convergence is $(-1, +1)$: both end points are excluded.

(3) The series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1}$$

has $R = 1$, the interval of convergence is $[-1, +1]$: both end points are included, but there the convergence is non-absolute [Sec. 255].

(4) For the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$R = 1$, the interval of convergence is $(-1, +1]$: the left end point is not included, but at the right end point the series converges non-absolutely [Sec. 256].

(5) Finally, we consider the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2};$$

here also $R = 1$, and the interval of convergence is $[-1, +1]$, the series converges absolutely at the end points $\left(\text{in view of the convergence of the series } \sum_{1}^{\infty} \frac{1}{n^2}\right)$.

All that has been said also holds for the general series (2), except that the role of the point 0 is played by the point x_0: the interval of convergence extends from $x_0 - R$ to $x_0 + R$ (with the inclusion of the end points or not, as the case may be).

Remark. If we extend the reasoning to the series

$$\sum_{n=0}^{\infty} c_n z^n = c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n + \ldots, \tag{1a}$$

ordered by powers of a complex variable z, with complex coefficients [Sec. 254], then it turns out that the theorem is also valid for such series: for every series (1a) (if we exclude everywhere divergent series!) there exists a number R, $0 < R \leqslant +\infty$, such that for $|z| < R$ the series is absolutely convergent, and for $|z| > R$ it is divergent. But in the "complex plane" the points for which $|z| < R$ form a circle of radius R (with centre at the point $z = 0$); thus, here there is a circle of convergence (instead of an interval of convergence), from which arises the name "radius of convergence".

273. The continuity of the sum of a power series. Let the series (1) have the radius of convergence $R > 0$. First of all we can assert:

(1) Whatever positive number $r < R$ we take, the series (1) *will converge uniformly with respect to x in the closed interval* $[-r, r]$. In fact, because $r < R$, then when $x = r$ the series (1) converges absolutely, i.e. the positive series

$$\sum_{n=0}^{\infty} |a_n| \cdot r^n = |a_0| + |a_1| \cdot r + |a_2| \cdot r^2 + \ldots + |a_n| \cdot r^n + \ldots \tag{5}$$

converges. When $|x| \leqslant r$ the terms of the series (1) do not exceed, in their absolute values, the corresponding terms of this series, which thus plays the part of a "major" series, and by Weierstrass's test the series (1) converges uniformly for the given values of x.

Although the number r can be as near to R as desired, yet nevertheless it does not follow from the proof that the series converges uniformly in the interval $(-R, R)$: this is seen merely from the example of the progression in [Sec. 264, *(5)*].

Now, as a corollary of Theorem 1, Sec. 266, we obtain:

(2) The sum $f(x)$ of the power series (1) inside its interval of convergence is a continuous function of x.

Whatever value of $x = x_0$ we take inside the interval of convergence, i.e. between $-R$ and R, we can choose a number

$0 < r < R$ such that $|x_0| < r$. Applying the Theorem 1 mentioned in the interval $[-r, r]$, in view of *(1)*, we can establish the continuity of the function $f(x)$ in this interval, hence, in particular, at $x = x_0$.

[We call the reader's attention to the fact that we avoid applying Theorem 1 to the interval $(-R, R)$, where uniform convergence cannot be guaranteed.]

The continuity of the sum of a power series can be used in the proof of the theorem on the identity of power series (we call to mind the similar theorem for whole polynomials):

(3) If two power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

and

$$\sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n + \ldots$$

in the neighbourhood of the point $x = 0$† *have exactly the same sum, then these series are identical,* i.e. their corresponding coefficients are equal in pairs:

$$a_0 = b_0,\ a_1 = b_1,\ a_2 = b_2,\ \ldots,\ a_n = b_n,\ \ldots.$$

Setting $x = 0$ in the identity

$$a_0 + a_1 x + \ldots = b_0 + b_1 x + \ldots,$$

we immediately verify the equation $a_0 = b_0$. Rejecting these terms on both sides of the identity we have written down and dividing them by x (in this case we are obliged to assume that $x \neq 0$), we obtain a new identity

$$a_1 + a_2 x + \ldots = b_1 + b_2 x + \ldots.$$

It also holds in the neighbourhood of the point $x = 0$, excluding this point itself, so that without some reason we have no right to set $x = 0$ in this identity. However, the continuity of the sums of

† Here we have in mind not only the two-sided neighbourhood $(-\delta, \delta)$ of the point $x = 0$, but also a one-sided neighbourhood of the form $[0, \delta)$ or $(-\delta, 0]$.

both the series as $x = 0$ enables us to conclude that the identity nevertheless is also satisfied at the point excluded, but then—letting $x = 0$—we obtain that $a_1 = b_1$. Rejecting these terms, dividing by $x \neq 0$ and again relying on the continuity (and only that), we find that $a_2 = b_2$, etc. The more general assertion is proved by means of mathematical induction.

This theorem, which establishes the uniqueness of the expansion of a function in a power series, was first mentioned by Euler.

274. Continuity at the end points of the interval of convergence. Now we consider the more subtle question of the behaviour of the power series (1) near one of the end points $x = \pm R$ of its interval of convergence (assuming R finite). Here we can confine ourselves to the right end point $x = R$; all that is said about it can be applied to the case of the left end point $x = -R$ by simply replacing x by $-x$.

In the form of a supplement to Theorem *2*, of the previous section, on the continuity of the sum of a power series in the open interval $(-R, R)$, we have

(4) ABEL'S THEOREM†. *If the power series* (1) *converges (even non-absolutely) when* $x = R$, *then its sum* $f(x)$ *for this value is continuous on the left, i.e.*

$$f(R-0) = \lim_{x \to R-0} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n R^n.$$

For the proof, one could establish that the hypothesis of the theorem implies the uniform convergence of the series (1) in the closed interval $[0, R]$, and then apply the general Theorem 1, Sec. 266. We prefer to give a direct proof.

Without loss of generality we can assume $R = 1$ (in this case the problem is reduced by replacing x by Rx). Thus, knowing that the series

$$\sum_{n=0}^{\infty} a_n = A$$

converges, we need to prove that

$$\lim_{x \to 1-0} \sum_{n=0}^{\infty} a_n x^n = A.$$

For this purpose, assuming from now on that $0 < x < 1$, we multiply the series (1) by the progression

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

† Niels Henrik Abel (1802–1829) was a well-known Norwegian mathematician.

by Cauchy's rule; we obtain

$$\frac{1}{1-x}\cdot\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}A_nx^n\,,\qquad\text{where}\qquad A_n=a_0+a_1+\ldots+a_n.$$

Now we subtract term by term the series

$$A=(1-x)\sum_{n=0}^{\infty}Ax^n$$

and

$$\sum_{n=0}^{\infty}a_nx^n=(1-x)\sum_{n=0}^{\infty}A_nx^n.$$

Letting $A-A_n=\alpha_n$, we arrive at the identity

$$A-\sum_{n=0}^{\infty}a_nx^n=(1-x)\sum_{n=0}^{\infty}\alpha_nx^n. \tag{6}$$

Because $\alpha_n\to 0$ [Sec. 235, *(2)*], then for an arbitrary given $\varepsilon>0$ we can find a number N such that $|\alpha_n|<1/2\varepsilon$, provided only $n>N$.

We separate the sum of the series on the right-hand side of (6) into the two sums

$$(1-x)\sum_{n=0}^{N}\alpha_nx^n\qquad\text{and}\qquad(1-x)\sum_{n=N+1}^{\infty}\alpha_nx^n.$$

The second can be calculated at once and is independent of x:

$$\left|(1-x)\sum_{n=N+1}^{\infty}\alpha_nx^n\right|\leqslant(1-x)\sum_{n=N+1}^{\infty}|\alpha_n|x^n<\frac{\varepsilon}{2}\cdot(1-x)\sum_{n=N+1}^{\infty}x^n<\frac{\varepsilon}{2}.$$

As far as the first is concerned, it tends to 0 as $x\to 1$, and for x sufficiently near 1 it will be true that

$$\left|(1-x)\sum_{n=0}^{N}\alpha_nx^n\right|<\frac{\varepsilon}{2},$$

so that finally

$$\left|A-\sum_{n=1}^{\infty}a_nx^n\right|<\varepsilon,$$

which proves the given assertion.

From this follows the simple

COROLLARY. *If, for the function $f(x)$, an expansion in a power series is obtained only in an open interval*

$$f(x)=\sum_{n=0}^{\infty}a_nx^n\qquad(-R<x<R),$$

but the function is defined and continuous, and the series continues to converge up to the end of this interval, say, where $x = R$, then the expansion remains valid also for $x = R$.

It is easy to ascertain this by taking limits in the given equation as $x \to R - 0$.

275. Termwise integration of a power series. With respect to integration and differentiation, power series—within the limits of their intervals of convergence—behave like ordinary polynomials.

(5) The power series (1) *in the interval* $[0, x]$, *where* $|x| < R$, *can always be integrated termwise, so that*—denoting by $f(x)$ the sum of the series:

$$\int_0^x f(x)\,dx = a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \ldots + \frac{a_{n-1}}{n}x^n + \ldots. \qquad (7)$$

For the proof we take r between $|x|$ and R. By virtue of *(1)*, Sec. 274, the series (1) converges uniformly in the interval $[-r, r]$, but then by Theorem 4, Sec. 269 the series can be integrated termwise in the interval $[0, x]$.

We will illustrate the various applications of this theorem by means of examples.

{1} By the termwise integration of the progressions

$$\frac{1}{1+x} = 1 - x + x^2 - \ldots + (-1)^{n-1}x^{n-1} + \ldots$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \ldots + (-1)^{n-1}x^{2(n-1)} + \ldots$$

in the interval $[0, x]$, where $|x| < 1$, we immediately obtain the expansions

$$\int_0^x \frac{dx}{1+x} = \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots,$$

$$\int_0^x \frac{dx}{1+x^2} = \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \ldots,$$

which were established in Secs. 257 and 256 by a considerably more complicated method. Because the first series converges when $x = 1$, and the second when $x = \pm 1$, the corresponding expansions (by the corollary to Abel's theorem) also hold for these values.

{2} Let us take the expansion of the function $(1+x)^{-1/2}$ in a series which is known to us [Sec. 258] and replace x by $-x^2$ in it (assuming $|x| < 1$); as a result we find that

$$\frac{1}{\sqrt{(1-x^2)}} = 1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{2n!!}x^{2n} \qquad (-1 < x < 1).$$

Now we integrate this series termwise in the interval $[0, x]$ $(-1 < x < 1)$; finally we obtain an expansion for the inverse sine which is new to us:

$$\int_0^x \frac{dx}{\sqrt{(1-x^2)}} = \arcsin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \ldots + \frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot \ldots \cdot 2n}\frac{x^{2n+1}}{2n+1} + \ldots .$$

By the corollary to Abel's theorem [Sec. 274], this expansion also holds for the end points $x = \pm 1$ of the interval, in so far as the series on the right also converges at these points [see, for instance, Sec. 240].

{3} A special part is played by termwise integration in the expansion in infinite power series of many integrals which cannot be expressed by elementary functions in a finite form [Sec. 165]. For instance, starting from the known expansion

$$e^{-x^2} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \ldots + (-1)^n \frac{x^{2n}}{n!} + \ldots$$

[Sec. 253, (11)], we find

$$\int_0^x e^{-x^2} dx = x - \frac{1}{1!}\frac{x^3}{3} + \frac{1}{2!}\frac{x^5}{5} - \ldots + (-1)^n \frac{1}{n!}\frac{x^{2n+1}}{2n+1} + \ldots .$$

Similar expansions can successfully be used for approximate calculations and for forming tables of values for integrals not expressible in finite form.

276. Termwise differentiation of a power series *(6) The power series* (1) *within its interval of convergence can be differentiated termwise*, so that for the sum $f(x)$ of the series there exists a derivative and can be expressed thus:

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + \ldots + n a_n x^{n-1} + \ldots . \qquad (8)$$

Whatever value of $x = x_0$, $-R < x_0 < R$, we take, we can choose two numbers r_0 and r such that $|x_0| < r_0 < r < R$. In view of the convergence of the series (5) [Sec. 273], its general term is bounded:

$$|a_n| \cdot r^n \leqslant L \quad (n = 1, 2, 3, \ldots ; L = \text{const}).$$

Then, when $|x| \leqslant r_0$,

$$n|a_n x^{n-1}| \leqslant n|a_n| \cdot r_0^{n-1} = n|a_n| \cdot r^{n-1} \cdot \left(\frac{r_0}{r}\right)^{n-1} \leqslant L_0 \cdot n \left(\frac{r_0}{r}\right)^{n-1}$$

$$\left(n = 1, 2, 3, \ldots ;\ L_0 = \frac{L}{r} = \text{const}\right)$$

so that the terms of the series (8), for the given values of x, do not exceed the corresponding terms of the "major" series

$$L_0 + L_0 \cdot 2\left(\frac{r_0}{r}\right) + L_0 \cdot 3\left(\frac{r_0}{r}\right)^2 + \ldots + L_0 \cdot n\left(\frac{r_0}{r}\right)^{n-1} + \ldots,$$

the convergence of which is easy to ascertain with the help of d'Alembert's test (taking into account that $r_0/r < 1$). Therefore in the interval $[-r_0, r_0]$ the series (8), consisting of the derivatives of the terms of the series (1), converges uniformly, and—by Theorem 5, Sec. 270—termwise differentiation of the series (1) is valid for the whole interval $[-r_0, r_0]$ and, in particular, for the point $x = x_0$.

Remark. We proved in *(5)* and *(6)* that the series (7) and (8) converge in the open interval $(-R, R)$ hence, their radii of convergence are not less than R. But in fact, the series (1) is obtained, in turn, by the termwise differentiation of (7) and the termwise integration of (8), so that R cannot be less than the radii of convergence given. From everything which has been said it follows that *the radii of all three series* (1), (7), (8) *are equal to each other.*

Example. In order to illustrate Theorem *(6)*, we consider the differential equation

$$xu'' + u' + xu = 0$$

(the simplest case of the so-called Bessel† equation, often encountered in mathematical physics and its applications). We set ourselves the problem of finding a solution u, which can be expanded as a series for all x.

We write the expansion of the function we are looking for in the form of a series with undetermined coefficients:

$$u = \sum_{n=0}^{\infty} a_n x^n$$

and, assuming that it converges everywhere, we differentiate it twice termwise, so,

$$xu = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=2}^{\infty} a_{n-2} x^{n-1}$$

$$u' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$xu'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1}.$$

† Friedrich Wilhelm Bessel (1784–1846) was a German astronomer.

Substituting these in the equation we arrive at the identity

$$a_1 + \sum_{n=2}^{\infty} (a_{n-2} + n^2 \cdot a_n) x^{n-1} = 0,$$

but then, by Theorem *(3)*, Sec. 273,

$$a_1 = 0, \quad n^2 \cdot a_n + a_{n-2} = 0 \quad (n = 2, 3, \ldots).$$

Hence, first of all, by induction we deduce that all the odd coefficients are zero, $a_{2m-1} = 0\,(m = 1, 2, 3, \ldots)$, but that as far as the coefficients of even value a_{2m} are concerned, by means of the recurrence formula

$$a_{2m} = -\frac{1}{4m^2} a_{2m-2}$$

they can be expressed in terms of a_0:

$$a_{2m} = (-1)^m \frac{1}{(m!)^2 \cdot 2^{2m}} a_0.$$

Thus, to within the arbitrary factor a_0, we have finally obtained the series

$$\sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{(m!)^2 \cdot 2^{2m}}.$$

It is easy to verify immediately that this series in fact converges everywhere. From the method by which it was obtained it is clear that the function† representing it satisfies the equation.

We draw the reader's attention to the original use of the *method of undetermined coefficients*: here there is an infinite set of these coefficients, and we were able to have recourse to the theorem on the identity of power series, instead of the usual application of the theorem on the identity of polynomials.

277. Power series as Taylor series. The previous theorem, *(6)*, suggested the possibility of successively repeated differentiation of a power series. Thus, as before, denoting by $f(x)$ the function representing the power series (1) in its interval of convergence, we will have everywhere inside this interval:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$
$$f'(x) = 1 \cdot a_1 + 2 \cdot a_2 x + 3 \cdot a_3 x^2 + \ldots + n \cdot a_n x^{n-1} + \ldots$$
$$f''(x) = 1 \cdot 2 \cdot a_2 + 2 \cdot 3 \cdot a_3 x + \ldots + (n-1) \cdot n \cdot a_n x^{n-2} + \ldots$$
$$f'''(x) = 1 \cdot 2 \cdot 3 \cdot a_3 + \ldots + (n-2) \cdot (n-1) \cdot n \cdot a_n x^{n-3} + \ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f^{(n)}(x) = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n a_n + \ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

† This is "the Bessel function of zero order"; it is denoted by $J_0(x)$.

If in all these equations we set $x = 0$, then we arrive at the expressions for the coefficients of a power series which are well known to us:

$$a_0 = f(0), \quad a_1 = f'(0), \quad a_2 = \frac{f''(0)}{2!}, \quad a_3 = \frac{f'''(0)}{3!}, \ldots, \quad a_n = \frac{f^{(n)}(0)}{n!}, \ldots$$

[cf. Sec. 252, (7)]. If we were discussing a series of general form (2) [Sec. 272], then here we need only substitute, instead of $x = 0$, $x = x_0$. Thus:

(7) The function representing a power series in its interval of convergence has derivatives of all orders inside this interval. The series itself, with respect to this function, is none other than its Taylor series.

This remarkable assertion throws light on the question of the expansion of a function in a power series, with which we were concerned in the preceding chapter. We saw that if a function can, in general, be expanded as a power series, then it must necessarily be the Taylor series; therefore we limit our investigation to the possibility that a function may be represented in a Taylor series.

278. The expansion of a continuous function in a series of polynomials. The class of functions which admit an expansion in a power series is extremely limited. Theorem *(7)* of the preceding section states that a function which can be expanded in a power series in any interval certainly must have derivatives of all orders here; yet even this strict condition, as we know from § 6, Chapter 15 [in particular see Sec. 259!], far from ensures the possibility of a power expansion.

In this connection the theorem proved by Weierstrass (in 1885) is important; for an arbitrary continuous function it establishes the possibility of an expansion in a uniformly convergent series, consisting of polynomials. We formulate it in the terminology of sequences:

WEIERSTRASS'S THEOREM. *If the function $f(x)$ is continuous in the finite closed interval $[a, b]$, then there exists a sequence of whole polynomials $\{P_n(x)\}$, which in this interval converges uniformly to $f(x)$.*

We presuppose first that we are discussing the interval $[0, 1]$. The required theorem is satisfied by the sequence of polynomials

$$B_n(x) = \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right) C_n^{\nu} x^{\nu} (1-x)^{n-\nu} \dagger. \tag{9}$$

† We reproduce the simplest proof of Weierstrass's theorem, which is due to the Soviet academician S. N. Bernstein. The polynomials $B_n(x)$ given, which are easily constructed for the values of the function $f(x)$ at the rational points ν/n, have been given the name "Bernstein polynomials" for the function $f(x)$.

To prove this we shall need a series of simple identities. First of all, for any natural n,

$$\sum_{\nu=0}^{n} C_n^\nu x^\nu (1-x)^{n-\nu} = 1, \tag{10}$$

which is obtained at once from the expansion of $(a+b)^n$ by Newton's binomial formula, if we take $a = x, b = 1-x$. Further,

$$\sum_{\nu=0}^{n} \nu C_n^\nu x^\nu (1-x)^{n-\nu} = nx. \tag{11}$$

In fact, rejecting the summand corresponding to $\nu = 0$, each of the remaining ones can be written in the form

$$nx \cdot \frac{(n-1)\cdot(n-2)\cdot \ldots \cdot (\overline{n-1} - \overline{\nu-1} + 1)}{1\cdot 2\cdot \ldots \cdot (\nu-1)} x^{\nu-1}(1-x)^{\overline{n-1}-\overline{\nu-1}}$$

or, if we introduce the symbol $\mu = \nu - 1$,

$$nx \cdot C_{n-1}^{\mu} x^\mu (1-x)^{\overline{n-1}-\mu}.$$

Taking nx outside the brackets, inside the brackets we obtain the sum

$$\sum_{\mu=0}^{n-1} C_{n-1}^{\mu} x^\mu (1-x)^{\overline{n-1}-\mu},$$

equal to 1 by virtue of (10). Analogously,

$$\sum_{\nu=0}^{n} \nu(\nu-1) C_n^\nu x^\nu (1-x)^{n-\nu} = n(n-1)x^2 \cdot \sum_{\lambda=0}^{n-2} C_{n-2}^{\lambda} x^\lambda (1-x)^{\overline{n-2}-\lambda}$$

$$= n(n-1)x^2. \tag{12}$$

Finally, multiplying (10) by n^2x^2, and (11) by $-(2nx-1)$, we add together these identities term by term with (12). As a result we arrive at the identity

$$\sum_{\nu=0}^{n} [n^2x^2 - (2nx-1)\nu + \nu(\nu-1)] C_n^\nu x^\nu (1-x)^{n-\nu}$$

$$= n^2x^2 - (2nx-1)nx + n(n-1)x^2$$

or, after simplification,

$$\sum_{\nu=0}^{n} (\nu - nx)^2 C_n^\nu x^\nu (1-x)^{n-\nu} = nx(1-x).$$

If we take into consideration that, for any x,

$$x(1-x) = \frac{1}{4} - \left(\frac{1}{2} - x\right)^2 \leqslant \frac{1}{4},$$

then we obtain the inequality

$$\sum_{\nu=0}^{n} (\nu - nx)^2 C_n^\nu x^\nu (1-x)^{n-\nu} \leqslant \frac{n}{4}, \tag{13}$$

which we can use immediately.

We fix x arbitrarily in the interval $[0, 1]$. In view of (10), we can write

$$f(x) = \sum_{\nu=0}^{n} f(x) C_n^\nu x^\nu (1-x)^{n-\nu}.$$

We subtract the last equation term by term from (9):

$$B_n(x) - f(x) = \sum_{\nu=0}^{n} \left[f\left(\frac{\nu}{n}\right) - f(x) \right] C_n^\nu x^\nu (1-x)^{n-\nu}. \tag{14}$$

To evaluate this difference we consider separately the summands corresponding to the points ν/n near to x, and the other summands. To be more precise, for a given number $\varepsilon > 0$—in view of the continuity, and hence [Sec. 75] the uniform continuity of the function $f(x)$—we can find a number $\delta > 0$, depending only on ε, such that

$$\text{from } |x'' - x'| < \delta \text{ it follows that } |f(x'') - f(x')| < \varepsilon$$
$$(0 \leqslant x', x'' \leqslant 1).$$

Here we separate the sum in (14) into two parts, Σ_1 and Σ_2, including in the first the summands for which $|\nu/n - x| < \delta$, and in the second those for which $|\nu/n - x| \geqslant \delta$. In the first case, clearly, $|f(\nu/n) - f(x)| < \varepsilon$ and [see (10)]

$$|\Sigma_1| < \varepsilon \sum_{\left|\frac{\nu}{n} - x\right| < \delta} C_n^\nu x^\nu (1-x)^{n-\nu} \leqslant \varepsilon \sum_{\nu=0}^{n} C_n^\nu x^\nu (1-x)^{n-\nu} = \varepsilon.$$

However in the second, introducing the greatest value M for $|f(x)|$, we have the approximation

$$|\Sigma_2| < 2M \sum_{\left|\frac{\nu}{n} - x\right| \geqslant \delta} C_n^\nu x^\nu (1-x)^{n-\nu}.$$

But here $(\nu - nx)^2 \geqslant \delta^2 n^2$, so that certainly

$$|\Sigma_2| < \frac{2M}{\delta^2 n^2} \sum_{\left|\frac{\nu}{n} - x\right| \geqslant \delta} (\nu - nx)^2 C_n^\nu x^\nu (1-x)^{n-\nu}$$

$$\leqslant \frac{2M}{\delta^2 n^2} \sum_{\nu=0}^{n} (\nu - nx)^2 C_n^\nu x^\nu (1-x)^{n-\nu}.$$

Finally, using the inequality (13), we obtain

$$\left|\sum\nolimits_2\right| < \frac{M}{2\delta^2 n}.$$

If we take $n > M/(2\delta^2\varepsilon)$, then (independently of x!) it will be true that

$$\left|\sum\nolimits_2\right| < \varepsilon,$$

and so, altogether,

$$|B_n(x) - f(x)| < 2\varepsilon,$$

which proves the required assertion—at least for the interval $[0, 1]$.

The case of an arbitrary interval $[a, b]$ reduces to the case considered by a simple change of variable: $x = a + y(b-a)$ where $0 \leqslant y \leqslant 1$. The sequence of polynomials $B_n(y)$ constructed for the continuous function of y: $f(a + y(b-a))$ converges to it uniformly for y in $[0, 1]$. Then the sequence of polynomials $P_n(x) = B_n\{(x-a)/(b-a)\}$ will converge to the function $f(x)$ uniformly for x in $[a, b]$. The theorem has been proved.

Going over in the usual way from a sequence of functions to series of functions [see Secs. 263 and 264], we can state Weierstrass's theorem in the following form:

Every function $f(x)$ continuous in the interval $[a, b]$ can be expanded into a series consisting of whole polynomials†

$$f(x) = \sum_{n=1}^{\infty} p_n(x),$$

which is uniformly convergent in this interval.

In actual applications, of course, such an expansion gives way to the expansion in a Taylor series, but, thanks to its extraordinary generality, it has great theoretical value. In particular, it follows from this that *every continuous function can be given by a unique "analytical expression"* [Sec. 18; cf. Euler's point of view, Sec. 21].

§ 4. An outline of the history of series

279. The epoch of Newton and Leibniz. Parallel with the discovery in the seventeenth century of differential and integral calculus, infinite series also came into use in mathematics. We begin with the work which appeared in England in 1668, which was more or less concerned with logarithmic series. In this year a short work was published by Nicholas Mercator (1620–1687), *Logarithmic Technique*, devoted to the methods of calculating logarithms. In one of the final chapters the author studies the rectangular hyperbola $xy = 1$, refers to its asymptotes. Letting $x = 1 + a$, he writes the equation of the hyperbola in the form

$$y = \frac{1}{1+a}.$$

† For instance, we can set $p_1(x) = P_1(x)$, $p_n(x) = P_n(x) - P_{n-1}(x)$ (for $n > 1$).

By division, Mercator turns this into an expansion in a geometric progression

$$\frac{1}{1+a} = 1 - a + a^2 - a^3 + \ldots,$$

and then, integrating termwise with respect to a, he obtains for the area of the hyperbola the well-known series [Sec. 256, (21)]

$$a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \ldots;$$

he does not touch on the problem of the domain in which the series applies (in the diagram supplied it happens that $a > 1$, so that the expansion is in fact not applicable!). The solution of this problem forms the basis of Wallis's revision. Finally, William Brounker (1620–1684) gave a purely geometric derivation of the special formula

$$\log 2 = \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \ldots.$$

For practical calculation he used a more rapidly converging series, where—by means of comparison with the geometric series—he found a bound for the remainder.

Somewhat earlier Newton began to study infinite series, but his research in this field, interwoven with his research in other fields of analysis, was published, as we know, only after a long delay. He first of all referred to this in letters in 1676, destined for Leibniz, and in his two fundamental works: *Analysis by Means of Equations with an Infinite Number of Terms* and *The Method of Fluxions and Infinite Series*, which were discussed in Chapter 14.

Apparently not later than 1666 Newton had already discovered the binomial series for fractional or negative values of the exponent. At first he arrived at it by inferences by analogy; the expansion of roots was verified by raising them to the appropriate power. Afterwards Newton found a direct method of dividing power series and extracting the roots of series, by applying to this the principles of the theory of decimal fractions, which was already known at the time. Expanding the different expressions into power series, Newton reduced the finding of their fluctions and fluents to such operations on powers, and by this means he greatly extended the domain of applicability of the analysis created by him.

Newton often had recourse to inverse series, i.e. starting from an expansion of one quantity in powers of another, he would establish an expansion of the second in terms of the first. Thus, starting from the logarithmic series

$$z = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots,$$

he obtained

$$x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \ldots,$$

i.e. essentially—the exponential series (adding one to each side we would obtain on the left the number $1+x$, the natural logarithm of which is exactly equal to z). It is interesting that the expansion of sin x by powers of its inverse, z†,

$$x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \ldots,$$

Newton found by an inversion of the series expressing the inverse

$$z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \ldots,$$

i.e. from the expansion of the inverse sine, which naturally seemed simpler to Newton, because it is obtained by integrating the function arc sin $(1-x^2)^{-1/2}$, which is easily expanded in a binomial series [Sec. 275, {2}].

Infinite series were widely used by Newton for the solution of equations, both algebraic and differential. The method he applied to them was essentially the method of undetermined coefficients.

When presenting in his *Analysis* the problem of solving equations with the help of series, Newton established the proof of the convergence of a series to the root of an equation. But in other cases, when expanding quantities, known in advance, in series, he did not study the convergence. However, because Newton saw the principal use of series as that for approximate calculations, naturally he was not interested in establishing formally the fact of the convergence of a series—what he needed was rapid convergence! For instance, Newton did not evaluate log 2 by means of the slowly converging series

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots,$$

but started from the formula $\log 2 = 2\log 1.2 - \log 0.8 - \log 0.9$: the arguments 1.2, 0.8, 0.9 differ little from 1, which ensures the rapid convergence of the logarithmic series. Newton did not estimate the error allowed.

Leibniz, independently of Newton, arrived at some of the expansions which Newton had discovered earlier. In 1693 Leibniz suggested for the integration of differential equations precisely the method of undetermined coefficients ("presupposing the desired series, as it were, already found"). In particular, in this way he again obtained expansions for logarithm and sign, starting from the differential equations which they satisfy.

In 1682 Leibniz first published the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots,$$

expressing the number $\pi/4$; Leibniz had discovered this series some time before‡.

† The radius of the circle is taken to be one.

‡ But apparently all this was anticipated by the Englishman James Gregory (1638–1675), who in a letter in 1671 communicated a series for the inverse tangent.

At the same time Leibniz gave an estimate for the error when the number $\pi/4$ is replaced by a segment of the series. These indications were afterwards generalized by him to the case of any alternating series whose terms decrease in absolute value to zero (for instance, in a letter to Johann Bernoulli, 1714). Here, however, the existence of a sum is self-understood, and what is established is in fact only the unlimited approximation to it of the partial sums, by turns from one side and then the other [cf. Sec. 244, Remark].

In later letters Leibniz spoke of series "converging" to their sums, meaning by convergence what we do. But at the same time he agreed with the assertion that the series $1-1+1-1+\ldots$ has as its sum the number $1/2$; this is obtained from the expansion

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$

when $x = 1$: if it is true when $x < 1$, then by the "law of continuity" it must be true for $x = 1$! Leibniz himself, it appears, was conscious of the lack of evidence; not without reason he said in one letter: "... it is only necessary to believe the reasoning on infinite series when it is possible to prove their truth with the help of finite quantities by Archimedes' method".

Two fellow workers of Leibniz, the brothers Bernoulli, also investigated infinite series, particularly the elder of them, Jacob: the collection of his works on series (1689–1704) gives an account of all that was known in this field in his time. In particular, first Johann, and then Jacob, gave a proof of the fact that "the sum of the infinite harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

is infinite". Jacob Bernoulli's proof was based on the principle which is usually used [Sec. 236, *(1)*]. He rightly emphasized in conclusion that "the sum of an infinite series whose last term vanishes, is sometimes finite and sometimes infinite". Of course, the "last term" must be interpreted as the limit of the general term. We must note that Jacob Bernoulli freely used divergent series and even obtained convergent expansions with their aid.

280. The period of the formal development of the theory of series. In the eighteenth century questions of principle did not attract much attention, but the use of series—mainly in the hands of Euler—made great progress.

At the beginning of the century, in 1715, there was published a small book by Taylor, *The Method of Differences, Direct and Inverse*. As a result of the obscurity of the presentation it did not at once become widespread. Taylor started from the consideration of finite differences and only then, going over, as a limit case, to infinitely small differences and their ratios, established an expansion of the increased value of the function x of z by powers of the increment v in the quantity z:

$$x + \frac{v}{1}\frac{dx}{dz} + \frac{v^2}{1\cdot 2}\frac{d^2x}{dz^2} + \frac{v^3}{1\cdot 2\cdot 3}\frac{d^3x}{dz^3} + \ldots;$$

this expansion afterwards came to be called the Taylor series. The author himself, evidently, did not fully appreciate his own discovery: it is applied very little in the book.

The value of this formula was revealed only in the vast *Treatise on Fluxions* by Maclaurin, published in 1742. Maclaurin arrived at the same series by another means: he took as his basis an expansion in powers of x with undetermined coefficients, differentiated it repeatedly and, each time letting $z = 0$, successively determined the coefficients [cf. Sec. 277, *(7)*]. By way of an example he obtained the binomial series. In the derivation of expansions for other simple functions Maclaurin also used the differential equations which they satisfied.

In the *Treatise* we find two other things which deserve mention. First, Maclaurin clearly established (it is true, only in geometric form) the integral test for the convergence and divergence of positive series (afterwards Cauchy proved it analytically; see Sec. 241). Then he derived (independently of Euler, who had done this several years earlier) the well-known formula† connecting the summation of terms of a given form with the evaluation of their integral.

In 1730 there began a brilliant series of works by Euler on infinite series. He devoted numerous papers to them, which were published over more than half a century in the works of the St. Petersburg Academy of Sciences; considerable space is also allotted to them in Euler's well-known treatises on analysis. We will enumerate briefly Euler's achievements, although not in their chronological order.

Euler first derived, with the help of implied passages to limits, the exponential and logarithmic series, starting from the binomial series [cf. Sec. 268]. In exactly the same way he obtained the series for cosine and sine from the known formulae for $\cos nz$ and $\sin nz$. He compared an infinite power series with an ordinary polynomial and expanded it by its factors, and in this way arrived at a representation of sine and other functions in the form of infinite products. By extending the known rule for multiplying together binomials with identical first terms to the case where there is an infinite number of them, he obtained the remarkable formulae

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}, \quad 1+\frac{1}{2^4}+\frac{1}{3^4}+\ldots=\frac{\pi^4}{90}$$

and others similar to them. The expansions of $1/\sin z$ and $\cot z$ in simple fractions are also due to Euler [see below, Sec. 406, *(3)*, Remark].

Euler also considered series with complex terms; comparison of the series for $\sin x$, $\cos x$ and e^x led him to the well-known formulae connecting these functions [Sec. 254]—in his time Lagrange called this "one of the most beautiful discoveries in analysis of this century".

Euler did considerable work on the summation of series. As we mentioned, he arrived at the well-known summation formula before Maclaurin and returned

† It is called the Euler–Maclaurin formula. Unfortunately, there is no room for it in our course.

to it more than once. In particular, he applied it to the partial sum of the harmonic series [cf. Sec. 236, *(1)* and Sec. 238, *(4)*]. Euler made considerable comparisons between series and integrals; in this way he found and studied the important integrals which were afterwards given his name [§ 4, Chapter 18 is devoted to them].

Euler came into contact with Maclaurin on one other point: the same considerations which led Maclaurin to his integral test were applied by Euler to establishing the limits between which the sum of a series is contained.

Of the other problems which interested Euler, we will mention the transformation of series—for the purpose of improving their convergence, and the transformation of infinite products to series [see, for instance, Sec. 251, *(3)*]. Finally, which it is particularly important to note, we find in Euler's work various applications of series not only in analysis itself, but also in algebra, in number theory and in other fields.

However, Euler did not obtain all these truly remarkable results with any valid justification. Like the majority of his contemporaries, Euler referred to the question of convergence carelessly and used divergent series freely. For instance, by adding the expansions

$$\frac{n}{1-n} = n + n^2 + n^3 + \ldots \quad \text{and} \quad \frac{n}{n-1} = 1 + \frac{1}{n} + \frac{1}{n^2} + \ldots,$$

he deduced that the infinite series on both sides

$$\ldots + \frac{1}{n^2} + \frac{1}{n} + 1 + n + n^2 + n^3 + \ldots \dagger$$

is equal to 0. Analogously, starting from the equation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots,$$

Euler not only deduced from this, when $x = -1$, the series already studied by Leibniz:

$$1 - 1 + 1 - 1 + \ldots = \frac{1}{2},$$

but also tried to make sense of the equation

$$1 + 2 + 4 + 8 + \ldots = -1,$$

obtained when $x = 2$. However, Euler stated that for such divergent series, it is impossible to speak of the sums in the same way as of the sum of a convergent series, which one can approach arbitrarily, by successively adding together its terms. Euler assumed that "the sum of an infinite series is a finite expression, from the expansion of which this series arises"‡. In this, so to speak, "generalized"

† The reader, of course, will notice that there are no values of n at all for which both progressions converge at the same time.

‡ See A. Euler, *Differential Calculus*, § 111.

notion of a sum there is a grain of truth (and it touched on contemporary fundamental theory of divergent series).

D'Alembert rose up firmly against the use of divergent series: reasoning carried out with their help seemed to him "suspect", even if the results are consistent with truth. However, d'Alembert considered the concepts of convergence and divergence to be of a "local" character, being connected with whether the succeeding term is less or greater than the preceding term (in absolute value). Thus, in his terminology, a series can be convergent in a definite place and only later begin to diverge, and conversely [cf. Remark Sec. 239].

We will conclude with Lagrange—one of the greatest mathematicians of the period under discussion. First of all, we must mention the series which bears his name, which gives an expansion of the root $x = p$ of the equation $a - x + \varphi(x) = 0$, and further—any function $\psi(p)$ of this root.

In his work *The Theory of Analytical Functions* (1797) Lagrange made an attempt to free differential calculus "from the consideration of infinitely small or vanishing quantities, limits or fluxions" and to reduce it to "the algebraic analysis of finite quantities"†. In this attempt the starting point was in fact power series. Assuming that for the function $f(x)$ the expansion

$$f(x+i) = f(x) + pi + qi^2 + ri^3 + \ldots$$

holds, where $p, q, r, \ldots$ are functions of x, Lagrange directly determined the successive "derivatives of the function" (it is here this term first appeared) by means of the coefficients of the expansion, starting from the usual relations

$$p = f'(x),\ q = \frac{f''(x)}{2},\ r = \frac{f'''(x)}{2 \cdot 3},\ \ldots.$$

On this basis Lagrange succeeded in presenting not only analysis, but also its applications to geometry and mechanics. Nevertheless, this point of view, appearing—exactly at the beginning of a new period in the history of analysis—as a reaction against the failure of all previous attempts to give it a rigorous foundation, did not become widespread.

In the same work Lagrange established the well-known [Sec. 106] convenient form for the remainder term in Taylor's formula. However, for Lagrange this term was simply an expression for the remainder of a series, whose convergence is presupposed; its use was only to facilitate the calculation of the error caused by ignoring the remaining terms of the series.

Lagrange (like Euler and his other contemporaries) regarded convergence as the tending to zero of the general term. He gave no definition of the concept of the sum of a convergent series, although it occurred more than once in a form sufficiently near our own, for instance, in the work of Maclaurin (1742) and of Waring (1776). At the beginning of the nineteenth century the French mathematician Fourier, in his well-known work *Analytical Theory of Heat* (1811; published in 1822), gave a valid definition of the convergence of a series and its sum,

† However, Lagrange had expressed the same idea earlier.

here that for the convergence of a series it is not at all sufficient that the terms of the series should "decrease continuously" to zero. As we recall, this fact had been emphasized more than a hundred years earlier by Jacob Bernoulli—in the example of the harmonic series.

Thus, the epoch we have been discussing, extraordinarily rich in formal achievements in the field of the theory of series, contributed little to its logical justification.

281. The creation of a precise theory. Only in the following period—in the nineteenth century—did series become an object for study in their own right, and the critical revision of the fundamentals of analysis in its early stages related to them.

In 1813 a work by Gauss was published† *General Investigations concerning the Series* $1+\frac{\alpha\cdot\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^2+\ldots.$ (This series was called the "hypergeometric" series.) Although in his concepts of convergence and divergence Gauss was apparently near those of d'Alembert, yet nevertheless it is here that we first find a precise formulation of the test which, for the series mentioned, gives a complete solution to the problem of convergence.

The contemporary definition of the concept of the sum of a series and its convergence or divergence, based on the concept of a limit, was finally established after the work of Bolzano (1817) and Cauchy (1821). In the preface to *Algebraic Analysis* Cauchy stated directly that "a divergent series has no sum". Moreover, both the works mentioned established in general form a condition necessary and sufficient for the convergence of a series [Sec. 242]. Cauchy also gave several simple and convenient tests, sufficient to establish the convergence or divergence of a positive series [Secs. 239, 241]. Afterwards there appeared a vast number of similar sufficiency tests, all more and more subtle and complicated‡; interest in this kind of investigation diminished at the beginning of our century.

Further, Cauchy made the assertion that the convergence of the series $\sum a_n$ follows from the convergence of the series $\sum |a_n|$ [Sec. 243], and distinguished between the cases of "absolute" and "non-absolute" convergence (although without using these terms). The actual fact of the existence of non-absolutely convergent series, e.g. the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots,$$

was already known in the time of Leibniz and the brothers Bernoulli, but Cauchy first drew attention to the fact that non-absolute convergence is reflected in the other properties of the series. First of all, as Cauchy showed in 1821, his theorem on the multiplication of series may not be applied to non-absolutely convergent

† Carl Friedrich Gauss (1777–1855)—a great German mathematician.

‡ Amongst Russian mathematicians N. I. Lobatchevsky and V. P. Yermakov (1845–1922) studied them.

series [Sec. 248, *(3)*]. Then, precisely in connection with the above-mentioned series, Cauchy noted (in 1833) that its convergence can be violated if the terms of the series are permuted.

In 1837 Dirichlet made the more general assertion that in an absolutely convergent series when the terms are permuted convergence and the value of the sum are preserved [Sec. 246]. At the same time he explained by means of examples that in the case of non-absolute convergence under such a permutation not only may the convergence be violated—this had been known before—but even when the convergence is preserved the value of the sum may be changed. The general and final result [Sec. 247]—in this direction—was established by Riemann; it was only published posthumously in 1867.

In *Algebraic Analysis* by Cauchy (1821) we find investigations relating to power series for the case not only of a real variable, but also of a complex variable. He not only established the form of the domain of convergence of such a series (the interval or circle of convergence), but also gave a precise expression for the radius of convergence in terms of the coefficients of the series. Inside the domain of convergence Cauchy dealt with power series as with ordinary polynomials, for instance he differentiated them termwise, without finding it necessary to give any justification of this.

With respect to general functional series Cauchy tried to prove the continuity of the sum of a convergent series of continuous functions (1821) and tried to integrate such a series termwise (1823), without any restrictions on the character of the convergence of the series [cf. Secs. 266 and 269]: Cauchy was not familiar with the concept of uniform convergence and, by using the language of infinitesimals, sometimes did not explain what variables they in fact depended on.

The erroneousness of Cauchy's first assertion was pointed out by Abel—in his famous memoranda, devoted to binomial series (1826). As a counter-example Abel quoted the sine series:

$$\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \ldots;$$

although it converges for all values of x, yet its sum at the points $x = (2m+1)\pi$ (m even) may suffer a discontinuity. At the same time for a power series Abel proved strictly the continuity of its sum for every value of the variable for which the series converges, even if this value coincides with an end point of the interval of convergence [Secs. 273, 274]. In his proof Abel in fact established the existence of exactly the property which afterwards was called uniform convergence.

Tchebychev expressed doubts (in 1845) about termwise integration and he emphasized that it is admissible "only in special cases". Thus there gradually grew up the conviction that it is impossible to extend the usual rules for finite sums without reservations to infinite functional series, although it was not clear in what cases this can nevertheless be done. A decisive part was played here by the study of the actual character of the convergence of functional series and by the introduction of the concept of uniform convergence.

This concept—and the theory itself—apparently first appeared in 1841 in a work (published only considerably later) by Weierstrass and was applied by him after that time in lectures. In print the distinction between uniform and non-uniform convergence was dealt with by Zaidel† in 1848 and Stokes‡ in 1849. If the sum of a series of continuous functions has a point of discontinuity, then near it "the series converges arbitrarily slowly" (Zaidel) or "the convergence of the series is infinitely retarded" (Stokes).

Here Stokes thought that, conversely, from the continuity of the sum of such a series one can deduce the absence of a retardation, i.e.—as we would say—the presence of uniform convergence; Zaidel considered this problem open. Afterwards it was proved by examples that the converse deduction is not in general true; its validity was proved by Dini in 1876 for the special case of a positive series [Sec. 267].

With this we finish our outline. A supplement to it can be found in "An outline of the history of trigonometrical series" at the end of Chapter 24 (see also Sec. 313, "Some historical remarks on changing the order of two limit operations").

† Philip Ludwig Zaidel (1821–1896)—a German mathematician.

‡ John Gabriel Stokes (1819–1903)—an English physicist and mathematician.

CHAPTER 17

IMPROPER INTEGRALS

§ 1. Improper integrals with infinite limits

282. The definition of integrals with infinite limits. In Chapter 11 we studied the concept of definite integrals $\int_a^b f(x)dx$ for the case of a finite interval $[a, b]$ and a bounded function $f(x)$. The present chapter is devoted to a generalization of this concept in several directions. We begin by considering an integral extended over an infinite interval.

Let the function $f(x)$ be defined over the interval $[a, \infty)$, i.e. for $x \geqslant a$, and integrable over any finite portion of it $[a, A]$, so that the integral $\int_a^A f(x)dx$ has a meaning for any $A > a$.

The finite or infinite limit of this integral as $A \to \infty$ is called the (improper) integral of the function $f(x)$ over the interval from a to ∞ and it is denoted by the symbol

$$\int_a^\infty f(x)dx = \lim_{A \to \infty} \int_a^A f(x)dx. \tag{1}$$

In the case when there exists a finite limit, we say that the integral (1) converges, and the function $f(x)$ is said to be integrable in the infinite interval $[a, \infty]$. The only difference from the integral in the proper sense, or proper integral, studied previously is that the integral (1) defined is called improper†.

If the limit (1) is infinite or does not exist at all, then one says of the integral that it diverges.

† We recall that we have already encountered the concept of an improper integral in Sec. 241.

Analogously to (1), we define the *integral of the function* $f(x)$ *from* $-\infty$ *to* a:

$$\int_{-\infty}^{a} f(x)dx = \lim_{A' \to -\infty} \int_{A'}^{a} f(x)dx \quad (A' < a); \tag{2}$$

in the same way we define the *integral of the function* $f(x)$ *from* $-\infty$ to $+\infty$:

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{\substack{A' \to -\infty \\ A \to +\infty}} \int_{A'}^{A} f(x)dx. \tag{3}$$

Here we again keep the terminology introduced for the integral (1).

In the last case, taking any a we can set

$$\int_{A'}^{A} f(x)\,dx = \int_{A'}^{a} f(x)dx + \int_{a}^{A} f(x)dx,$$

and the existence of a limit as $A' \to -\infty$, $A \to +\infty$ for the integral on the left is clearly equivalent to the existence separately of the limits (1) and (2) for the integrals on the right. Thus, the integral from $-\infty$ to $+\infty$ can also be defined by the equation

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{+\infty} f(x)dx,$$

presupposing the existence separately of the integrals on the right†. This definition does not in fact depend on the choice of the point a.

Examples. (1) The function $1/(1+x^2)$ is integrable in any finite interval $[0, A]$ $(A > 0)$, where we have

$$\int_{0}^{A} \frac{dx}{1+x^2} = \arctan A.$$

Because as $A \to \infty$ there exists for this integral a finite limit $\pi/2$, the integral from 0 to ∞ converges and has the value

$$\int_{0}^{\infty} \frac{dx}{1+x^2} = \lim_{A \to \infty} \int_{0}^{A} = \frac{\pi}{2}.$$

† We exclude only the case when these integrals are equal to an infinity of different values.

$$(2)\quad \int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{A'\to-\infty} \int_{A'}^{0} \frac{dx}{1+x^2} = \lim_{A'\to-\infty} (-\arctan A') = \frac{\pi}{2}.$$

$$(3)\quad \int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{0}^{+\infty} + \int_{-\infty}^{0} = \pi.$$

283. The application of the fundamental formula of integral calculus. In the examples given above the integral was evaluated over a finite interval with the help of a primitive function, and then limits were taken. We can join together both operations in one formula.

Let, for instance, the function $f(x)$ be defined and continuous in the interval $[a, \infty)$, so that for $f(x)$ in this interval there exists a primitive function $F(x)$ [Sec. 183], and by the fundamental formula of integral calculus [Sec. 185]

$$\int_{a}^{A} f(x)\,dx = F(A) - F(a) = F(x)\Big|_{a}^{A}.$$

Hence it is clear that the improper integral (1) exists if and only if there exists a finite limit

$$\lim_{A\to\infty} F(A) = F(\infty),$$

and then

$$\int_{a}^{\infty} f(x)\,dx = F(\infty) - F(a) = F(x)\Big|_{a}^{\infty}.$$

Analogously,

$$\int_{-\infty}^{a} f(x)\,dx = F(x)\Big|_{-\infty}^{a}, \qquad \int_{-\infty}^{+\infty} f(x)\,dx = F(x)\Big|_{-\infty}^{+\infty},$$

if by $F(-\infty)$ we understand the limit $\lim_{A'\to-\infty} F(A')$. The actual possibility of making the double substitution is connected with the existence and finiteness of the limit appearing in it, and testifies the existence of the integral.

We turn to some further examples.

(4) We study first of all the problem of with what values of the exponent λ the improper integral

$$\int_{a}^{\infty} \frac{dx}{x^{\lambda}} \qquad (a > 0)$$

exists.

Let $\lambda \neq -1$, then

$$\int_a^\infty \frac{dx}{x^\lambda} = \frac{1}{1-\lambda} x^{1-\lambda}\Big|_a^\infty .$$

When $\lambda > 1$ the double substitution—and with it the integral—has a finite value; when $\lambda < 1$ both expressions become ∞. When $\lambda = 1$ we have

$$\int_a^\infty \frac{dx}{x} = \log x\Big|_a^\infty = \infty .$$

Thus, the given integral converges when $\lambda > 1$, and when $\lambda \leqslant 1$ it diverges.

(5) We consider the integrals $(a > 0)$

$$\int_0^\infty e^{-ax} \sin bx\, dx, \quad \int_0^\infty e^{-ax} \cos bx\, dx.$$

Recalling their respective primitives [Sec. 163, *(4)*], we obtain at once

$$\int_0^\infty e^{-ax} \sin bx\, dx = -\frac{a\sin bx + b\cos bx}{a^2+b^2} e^{-ax}\Big|_0^\infty = \frac{b}{a^2+b^2},$$

$$\int_0^\infty e^{-ax} \cos bx\, dx = \frac{b\sin bx - a\cos bx}{a^2+b^2} e^{-ax}\Big|_0^\infty = \frac{a}{a^2+b^2}.$$

(6) Finally, the integrals

$$\int_0^\infty \sin x\, dx, \quad \int_0^\infty \cos x\, dx$$

do not exist, in so far as the double substitutions

$$-\cos x\Big|_0^\infty, \quad \sin x\Big|_0^\infty$$

have no meaning: $\cos x$ and $\sin x$ have no limits as $x \to \infty$.

284. An analogy with series. Some simple theorems. In what follows we limit ourselves to integrals of the form (1): all that is said about them can easily be extended to the cases (2) and (3). Here we will always presuppose that the function $f(x)$ is integrable in the proper sense between any limits a and $A > a$, so that the problem only relates to the improper integral from a to ∞.

Between improper integrals $\int_a^\infty f(x)\, dx$ and numerical series $\sum_1^\infty a_n$ there exists a strong analogy, which it is useful to emphasize.

If the process of summation with respect to n is replaced by integration with respect to x then the analogies will be

the general term	*the function under integration*
a_n	$f(x)$
the partial sum of the series	*the proper integral*
$\sum_1^N a_n$	$\int_a^A f(x)\,dx$
the sum of the series	*the improper integral*
$\sum_1^\infty a_n$	$\int_a^\infty f(x)\,dx$
as the limit of the partial sum as $N \to \infty$	*as the limit of the preceding integral as* $A \to \infty$
the remainder of the series	*the integral*
$\sum_{N+1}^\infty a_n$	$\int_A^\infty f(x)\,dx$

We will enumerate the simplest theorems on improper integrals; these resemble the theorems in Sec. 235 on series. Their proof—by means of the analogy indicated—we leave to the reader.

(1) If the integral $\int_a^\infty f(x)\,dx$ *converges, then the integral* $\int_A^\infty f(x)\,dx$ $(A > a)$ *also converges, and conversely. Here*

$$\int_a^\infty f(x)\,dx = \int_a^A f(x)\,dx + \int_A^\infty f(x)\,dx.$$

(2) In the case when the integral $\int_a^\infty f(x)\,dx$ *converges, we have*

$$\lim_{A \to \infty} \int_A^\infty f(x)\,dx = 0.$$

(3) From the convergence of the integral $\int_a^\infty f(x)\,dx$ *follows*

the convergence of the integral $\int_a^\infty c \cdot f(x)\,dx$ $(c = \text{const})$ *where*

$$\int_a^\infty c \cdot f(x)\,dx = c \cdot \int_a^\infty f(x)\,dx.$$

Finally:

(4) If both the integrals $\int_a^\infty f(x)\,dx$ *and* $\int_a^\infty g(x)\,dx$ *converge, then the integrals* $\int_a^\infty [f(x) \pm g(x)]\,dx$ *converge, and*

$$\int_a^\infty [f(x) \pm g(x)]\,dx = \int_a^\infty f(x)\,dx \pm \int_a^\infty g(x)\,dx.$$

285. The convergence of the integral in the case of a positive function. If the function $f(x)$ is positive (non-negative), then the integral

$$\Phi(A) = \int_a^A f(x)\,dx \tag{4}$$

is a monotonically increasing function of the variable A. The question of the existence of a finite limit for it as $A \to \infty$ is solved very simply by the fundamental theorem on the limit of a monotonic function [Sec. 47]:

For the convergence of the improper integral (1)*—in the case of a positive function* $f(x)$*—it is necessary and sufficient that as* A *increases the integral* (4) *remains bounded above:*

$$\int_a^A f(x)\,dx \leqslant L \qquad (L = \text{const}).$$

If however this condition is not satisfied, then the integral (1) *has the value* ∞. [Cf. Sec. 236.]

On this is based the following "comparison theorem" for the integrals of positive functions:

THEOREM 1. *If, for all* $x \geqslant A$ $(A \geqslant a)$, *the inequality* $f(x) \leqslant g(x)$ *holds, then from the convergence of the integral* $\int_a^\infty g(x)dx$ *follows the convergence of the integral* $\int_a^\infty f(x)\,dx$ *or, equivalently, from the divergence of* $\int_a^\infty f(x)\,dx$ *follows the divergence of* $\int_a^\infty g(x)\,dx$.

The proof can be modelled on the proof of Theorem 1, Sec. 237.

The following theorem, which is a corollary of the first, is often useful:

THEOREM 2. *If there exists a limit*

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=K \quad (0\leqslant K\leqslant+\infty),$$

then from the convergence of the integral $\int_a^\infty g(x)\,dx$, *when* $K<+\infty$, *follows the convergence of the integral* $\int_a^\infty f(x)\,dx$, *and from the divergence of the first integral, when* $K>0$, *follows the divergence of the second.* [Thus, when $0<K<+\infty$, both integrals converge or diverge together.]

Proof. This is the same as that for the analogous Theorem 2, Sec. 237.

By choosing an actual function for comparison, we can obtain from this a special test for the convergence or divergence of the integral $\int_a^\infty f(x)\,dx$. Comparison with the function $1/x^\lambda$ is of practical value; it is integrable from $a>0$ to ∞ when $\lambda>1$ and is not integrable when $\lambda\leqslant 1$ [Sec. 283, *(4)*]. Using this we construct the following tests.

For sufficiently large x, let the function f(x) have the form

$$f(x)=\frac{\varphi(x)}{x^\lambda} \qquad (\lambda>0).$$

Then: (1) if $\lambda>1$ *and* $\varphi(x)\leqslant c<+\infty$, *then the integral* $\int_a^\infty f(x)\,dx$ *converges, (2) if, however,* $\lambda\leqslant 1$ *and* $\varphi(x)\geqslant c>0$, *then this integral diverges.*

For the proof it is necessary to use Theorem 1; the comparison function is c/x^λ [Sec. 284, *(3)*].

If as $x\to\infty$ *the function f(x) is infinitely small of order* $\lambda>0$ *(by comparison with* $1/x$*), then the integral* $\int_a^\infty f(x)\,dx$ *converges or diverges according to whether* $\lambda>1$ *or* $\leqslant 1$.

Here it is necessary to appeal to Theorem 2; the part of the function $g(x)$ is played by $1/x^{\lambda}$.

Examples.

$$\int_0^{\infty} \frac{x^{3/2}}{1+x^2}\,dx, \qquad \int_1^{\infty} \frac{dx}{x\sqrt{(1+x^2)}}.$$

The expressions under the integrals as $x \to \infty$ are infinitely small, respectively of order 1/2 and 2. Hence, the first integral diverges, and the second converges.

286. The convergence of the integral in the general case. The question of the convergence of the improper integral $\int_a^{\infty} f(x)\,dx$, according to (1), can be reduced to the question of the existence of a finite limit as $A \to \infty$ for the function (4) of A. Applying the Bolzano–Cauchy theorem [Sec. 53] to this function, we can represent the condition for the convergence of the improper integral in the following form:

For the convergence of the improper integral $\int_0^{\infty} f(x)\,dx$ *it is necessary and sufficient that to every number* $\varepsilon > 0$ *there corresponds a number* $A_0 > a$, *such that when* $A > A_0$ *and* $A' > A$ *the inequality*

$$|\Phi(A') - \Phi(A)| = \left|\int_a^{A'} f(x)\,dx - \int_a^{A} f(x)\,dx\right| = \left|\int_A^{A'} f(x)\,dx\right| < \varepsilon$$

is satisfied.

This criterion enables us easily to establish this proposition:

If the integral $\int_a^{\infty} |f(x)|\,dx$ *converges, then certainly the integral* $\int_a^{\infty} f(x)\,dx$ *converges.*

In fact, applying the criterion given to the integral $\int_a^{\infty} |f(x)|\,dx$, which we presuppose to be convergent, we see that for any $\varepsilon > 0$ we can find a number $A_0 > a$ such that

$$\int_A^{A'} |f(x)|\,dx < \varepsilon,$$

provided only $A' > A > A_0$. But, clearly,

$$\left|\int_A^{A'} f(x)\,dx\right| \leqslant \int_A^{A'} |f(x)|\,dx$$

and, hence, for all such A, A' the inequality

$$\left|\int_{A}^{A'} f(x)\,dx\right| < \varepsilon$$

is satisfied hence, by virtue of our criterion, the convergence of the integral $\int_{a}^{\infty} f(x)\,dx$ follows.

We note that from the convergence of this latter integral, generally speaking, the convergence of the integral $\int_{a}^{\infty} |f(x)|\,dx$ does not follow. This circumstance causes us to distinguish in particular the following case. If together with the integral $\int_{a}^{\infty} f(x)\,dx$ the integral $\int_{a}^{\infty} |f(x)|\,dx$ also converges, then the integral $\int_{a}^{\infty} f(x)\,dx$ is said to be absolutely convergent, and the function $f(x)$ is said to be absolutely integrable in the interval $[a, \infty]$†. An example of an integral which converges non-absolutely will be given in Sec. 287.

The theorem proved enables one to establish the convergence of the integral of an alternating function $f(x)$, by applying the test of the preceding section to the positive function $|f(x)|$: if this function is shown to be integrable, then the function $f(x)$ will also be integrable and, moreover, absolutely.

Let us, for instance, consider the integral $\int_{0}^{\infty} \frac{\cos ax}{k^2 + x^2}\,dx$. Because

$$\left|\frac{\cos ax}{k^2 + x^2}\right| \leqslant \frac{1}{k^2 + x^2},$$

and the integral of the function on the right converges, then [Sec. 285, Theorem 1] the integral of the function on the left also converges, and with it the integral considered converges (absolutely). As one can see, the considerations given here for an alternating function—under favourable conditions—can only establish absolute convergence. If, on the other hand, the integral of the given function diverges or converges, but not absolutely, then it is impossible to distinguish these cases using just the tests established for positive functions.

† Again there is an analogy with infinite series [cf. Sec. 243].

287. More refined tests. We will now give some tests of a different type; these will enable us to establish the convergence of improper integrals in a number of cases when absolute convergence does not occur.

Let us have two functions $f(x)$ and $g(x)$, defined and continuous in the interval $[a, \infty)$; we will presuppose that $g(x)$ is monotonic and has a continuous derivative (clearly, not one with changing sign!). We are interested in a condition which will guarantee the convergence of the integral of the product

$$\int_a^\infty f(x)g(x)dx. \tag{5}$$

We presuppose that

(I) *(1) the integral* (4)

$$\Phi(A) = \int_a^A f(x)dx$$

is a bounded function of A:

$$|\Phi(A)| \leqslant K \quad (K = \text{const},\ a \leqslant A < \infty)$$

(even though the limit $\lim_{A\to\infty} \Phi(A)$, i.e. the improper integral $\int_a^\infty f(x)dx$, may not exist);

(2) $g(x) \to 0$ *as* $x \to \infty$.

Then the integral (5) *converges.*

For any $A' > A > a$, integrating by parts, we have:

$$\int_A^{A'} f(x)g(x)\,dx = \Phi(A')g(A') - \Phi(A)g(A) - \int_A^{A'} \Phi(x)g'(x)dx;$$

if we apply the generalized mean value theorem [Sec. 182, *(10)*] to the last integral, it can be represented in the form

$$\int_A^{A'} \Phi(x)\,g'(x)dx = \Phi(\xi)\int_A^{A'} g'(x)dx = \Phi(\xi)\,[g(A') - g(A)] \quad (A \leqslant \xi \leqslant A')$$

and finally

$$\int_A^{A'} f(x)g(x)dx = [\Phi(A') - \Phi(\xi)]g(A') + [\Phi(\xi) - \Phi(A)]g(A). \tag{6}$$

Because, by virtue of *(1)*, both brackets do not exceed $2K$ in absolute value and, in view of *(2)*, we can find an A_0 such that when $x > A_0$

$$|g(x)| < \varepsilon/4K$$

(where $\varepsilon > 0$ is chosen arbitrarily in advance), then when $A' > A > A_0$

$$\left|\int_A^{A'} f(x)g(x)\right| < \varepsilon,$$

whence follows the convergence of the integral (5) [Sec. 286].

We can replace these expressions for the conditions by others, which require more of the factor f, but on the other hand lighten the conditions imposed on the factor g. *In fact, now let*

(II) *(1)' there exist the improper integral*

$$\int_a^\infty f(x)\,dx = \lim_{A\to\infty} \Phi(A);$$

(2') the function $g(x)$ be bounded:

$$|g(x)| \leqslant L \quad (L = \text{const};\ a \leqslant x < \infty).$$

With these conditions the integral (5) again converges.

This time the second factor in (6) is bounded, and the first—with appropriate values for A and A'—can be made as small as desired, which leads to the same result.

We turn to some examples.

The integrals

$$\int_a^\infty \frac{\sin x}{x^\lambda}\,dx, \quad \int_a^\infty \frac{\cos x}{x^\lambda}\,dx$$

(when $a > 0$ and $\lambda > 0$) converge by the test (I). As the function $f(x)$ here we can take, respectively, $\sin x$ or $\cos x$, because

$$\left|\int_a^A \sin x\,dx\right| = |\cos a - \cos A| \leqslant 2, \quad \left|\int_a^A \cos x\,dx\right| \leqslant 2,$$

although in an infinite interval these functions are not integrable [Sec. 283, *(6)*]; the part of $g(x)$ is played by $1/x^\lambda$.

When $\lambda > 1$ these integrals converge absolutely, in so far as the integral

$$\int_a^\infty \frac{dx}{x^\lambda}$$

converges [Sec. 283, *(4)*], and $|\sin x|$ and $|\cos x| \leqslant 1$. On the contrary, when $\lambda \leqslant 1$ both integrals converge non-absolutely. In order to prove this, for instance, for the integral

$$\int_a^\infty \frac{\sin x}{x}\,dx,$$

it is necessary to establish that the integral

$$\int_a^\infty \frac{|\sin x|}{x}\,dx$$

diverges. In fact, if this integral were convergent, then, in view of the inequality

$$|\sin x| \geqslant \sin^2 x,$$

certainly [Sec. 285, Theorem 1] the integral

$$\int_a^\infty \frac{\sin^2 x}{x}\,dx = \frac{1}{2}\int_a^\infty \frac{1-\cos 2x}{x}\,dx$$

would converge; adding to it the known convergent integral

$$\frac{1}{2}\int_a^\infty \frac{\cos 2x}{x}\,dx,$$

we would be led to the conclusion that the integral

$$\frac{1}{2}\int_a^\infty \frac{dx}{x}$$

converges, which is in fact not so [Sec. 283, *(4)*].

Remark. Now, since we have established the convergence of the integrals

$$\int_a^\infty \frac{\sin x}{x}\,dx \quad \text{and} \quad \int_a^\infty \frac{\cos x}{x}\,dx,$$

we can, finally, make more precise the definition of the non-elementary functions si x ("the integral sine") and ci x ("the integral cosine"), which we mentioned in the first volume (p. 340). In fact, we put

$$\operatorname{si} x = -\int_{x\,(x\geqslant 0)}^\infty \frac{\sin t}{t}\,dt, \qquad \operatorname{ci} x = -\int_{x\,(x>0)}^\infty \frac{\cos t}{t}\,dt.$$

If, for instance, we write the second of these formulae in the form:

$$\operatorname{ci} x = \int_1^x \frac{\cos t}{t}\,dt - \int_1^\infty \frac{\cos t}{t}\,dt,$$

then—by a well-known property of a definite integral [Sec. 183, *(12)*]—it is clear that the derivative of ci x is in fact equal to $(\cos x)/x$.

§ 2. Improper integrals of unbounded functions

288. The definition of integrals of unbounded functions. Now we consider a function $f(x)$, given in a finite interval $[a, b]$, but not integrable in this interval. We presuppose more definitely that it is integrable in any interval $[a, b-\eta]$ where $0<\eta<b-a$, but becomes non-integrable in every interval $[b-\eta, b]$ on the left of the point b. The point b here is called a *singular point.*

We can show that near the point b the function $f(x)$ will then be unbounded, and this indeed is the cause here of the fact that near b it is non-integrable. The case we shall usually meet is when the function $f(x)$ "becomes infinite" at the point b (we recall that by this we simply mean that as $x \to b$ the function tends to infinity).

The finite or infinite limit of the integral $\int_a^{b-\eta} f(x)dx$ *as* $\eta \to 0$ *is called the (improper) integral of the function* $f(x)$ *in the interval from a to b* and we write:

$$\int_a^b f(x)\,dx = \lim_{\eta \to 0} \int_a^{b-\eta} f(x)\,dx. \tag{1}$$

In the case when a finite limit exists, we say that the integral (1) *converges*, and the function $f(x)$ is said to be *integrable* in the interval $[a, b]$. Otherwise, we say that the integral *diverges*.

Example. (1) The function $1/\sqrt{(1-x^2)}$ is bounded and integrable in any interval $[0, 1-\eta]$ $(0 < \eta < 1)$, and

$$\int_0^{1-\eta} \frac{dx}{\sqrt{(1-x^2)}} = \arcsin(1-\eta).$$

At the point $x = 1$ the function becomes infinite. Clearly, the point $x = 1$ is singular.

Because the integral given tends to the limit $\arcsin 1 = \pi/2$ as $\eta \to 0$, the improper integral

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)}} = \lim_{\eta \to 0} \int_0^{1-\eta} = \frac{\pi}{2}$$

exists.

Now let the function $f(x)$ be bounded and integrable in any interval $[a+\eta', b]$ $(0 < \eta' < b-a)$, but let it become non-integrable in every interval $[a, a+\eta']$ to the right of the point a (a singular point). Then *the (improper) integral of the function* $f(x)$ *from a to b is defined by the equation*

$$\int_a^b f(x)\,dx = \lim_{\eta' \to 0} \int_{a+\eta'}^b f(x)\,dx. \tag{2}$$

If both the points a and b are singular, then *the definition of the integral from a to b is given by the equation*

$$\int_a^b f(x)\,dx = \lim_{\substack{\eta_1 \to 0 \\ \eta_2 \to 0}} \int_{a+\eta_1}^{b-\eta_2} f(x)\,dx. \tag{3}$$

The definition (3) can be replaced by

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

with the hypothesis that $a < c < b$ and both of the improper integrals on the right exist (here the choice of the point c is arbitrary)†.

We keep the same terminology as above for the improper integrals (2) and (3).

Examples.

(2) $\int_{-1}^{0} \frac{dx}{\sqrt{(1-x^2)}}$, with the singular point -1.

$$\int_{-1}^{0} \frac{dx}{\sqrt{(1-x^2)}} = \lim_{\eta' \to 0} \int_{-1+\eta'}^{0} \frac{dx}{\sqrt{(1-x^2)}} = \lim_{\eta' \to 0} [-\arcsin(-1+\eta')] = \frac{\pi}{2};$$

(3) $\int_{-1}^{1} \frac{dx}{\sqrt{(1-x^2)}}$, with the two singular points -1 and 1,

$$\int_{-1}^{1} \frac{dx}{\sqrt{(1-x^2)}} = \int_{-1}^{0} + \int_{0}^{1} = \pi.$$

It is not difficult to see how to define the improper integral $\int_a^b f(x)dx$ when there exists any (finite) number of singular points.

Finally, we consider the function $f(x)$, given in an infinite interval, for instance, in $[a, \infty]$, and having a finite number of singular points in it‡, near which it ceases to be integrable. We presuppose that

† Cf. the footnote on p. 111.

‡ It is possible to have an infinite set of singular points, but in every finite interval $[a, A]$ $(A > a)$ there can only be a finite number of them (which may increase to infinity with A).

the integral $\int_a^A f(x)dx$ exists in every finite interval $[a, A]$, being a proper or improper integral according to the definition given above. Then, taking limits at once as $A \to \infty$, we can define by means of equation (1) [Sec. 282] the improper integral in the interval $[a, \infty]$.

289. An application of the fundamental formula of integral calculus. Let the function $f(x)$ be defined and continuous in the interval $[a, b)$, while at b it has a singularity. For x in the interval $[a, b)$ there exists a primitive function $F(x)$, and

$$\int_a^{b-\eta} f(x)\,dx = F(b-\eta) - F(a) = F(x)\Big|_a^{b-\eta},$$

so that the convergence of the improper integral (1) is equivalent to the existence of a finite limit $\lim_{\eta \to 0} F(b-\eta)$. If the latter exists, then it is natural to take it as the value $F(b)$ of the primitive function at $x = b$, thus arriving at the continuity of $F(x)$ in the whole interval $[a, b]$. For the calculation of the integral (1) we then have a formula of the usual form:

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b. \tag{4}$$

The same formula holds in the case when the singular point lies inside the interval or when there exist several singular points, but (and it is necessary to bear this firmly in mind) with the indispensable condition that the primitive function $F(x)$, having $f(x)$ as its derivative everywhere excluding the singular points, *should be continuous also at these latter points*. The existence of such a primitive guarantees the convergence of the improper integral. If, however, the primitive becomes infinite at an end point of the interval or inside it, then the integral diverges.

We turn to some examples.

(4) We will investigate for what values of the exponent $\lambda > 0$ the improper integral

$$\int_a^b \frac{dx}{(x-a)^\lambda} \qquad (b > a)$$

converges.

Presupposing that $\lambda \neq 1$, we have

$$\int_a^b \frac{dx}{(x-a)^\lambda} = \frac{1}{1-\lambda}(x-a)^{1-\lambda}\Big|_a^b,$$

so that the integral converges when $\lambda < 1$ and diverges when $\lambda > 1$: the primitive becomes infinite at the end point $x = a$. For the same reason the integral also diverges when $\lambda = 1$:

$$\int_a^b \frac{dx}{x-a} = \log(x-a)\Big|_a^b.$$

(5) An analogous result can be established for the integral

$$\int_a^b \frac{dx}{(b-x)^\lambda} \qquad (b > a,\ \lambda > 0),$$

which does not differ essentially from the preceding one.

(6) For the integral $\int_0^1 \log x\,dx$ there is the singular point $x = 0$. We have

$$\int_0^1 \log x\,dx = x\log x - x\Big|_0^1 = -1,$$

because as $x \to 0$ the primitive has the limiting value 0.

290. Conditions and tests for the convergence of an integral. We shall limit ourselves to the case given in the definition (1), because the rephrasing of our results for the other cases presents no difficulties. In view of the complete analogy with the improper integral considered in the infinite interval $[a, \infty]$, we will not enumerate the simplest theorems [cf. Sec. 284], and on the question of the existence of the integral we will limit ourselves to formulating several basic situations —their proof is analogous to those mentioned above. In all cases the integrability (in the proper sense) of the functions considered in any interval $[a, b-\eta]$ $(\eta > 0)$ is presupposed, and only the point $x = b$ is singular.

As in Sec. 285:

For the existence of the improper integral (1) when $f(x)$ is a positive function, it is necessary and sufficient that the integral $\int_a^{b-\eta} f(x)dx$ $(\eta > 0)$ should remain bounded above:

$$\int_a^{b-\eta} f(x)dx \leqslant L \qquad (L = \text{const}).$$

If, however, this condition is not satisfied, then the integral (I) has the value $+\infty$.

The comparison Theorems 1 and 2 in Sec. 285 for positive functions also hold here. We will not reproduce them, but will formulate some fundamental tests based on them.

For a value of x *sufficiently near* b *let the function* $f(x)$ *have the form:*

$$f(x) = \frac{\varphi(x)}{(b-x)^{\lambda}} \qquad (\lambda > 0).$$

Then: (1) if $\lambda < 1$ *and* $\varphi(x) \leqslant c < \infty$, *then the integral* $\int_a^b f(x)\,dx$ *converges, (2) if, however,* $\lambda \geqslant 1$ *and* $\varphi(x) \geqslant c > 0$, *then this integral diverges.*

A more special form, often convenient in practice:

If as $x \to b$ *the function* $f(x)$ *is infinitely large of order* $\lambda > 0$ *(by comparison with* $1/(b-x)$*), then the integral* $\int_a^b f(x)\,dx$ *converges or diverges according to whether* $\lambda < 1$ *or* $\lambda \geqslant 1$.

Examples.

(1) $\int_0^1 \frac{dx}{\sqrt[4]{(1-x^4)}}$.

As $x \to 1$ the function under the integral becomes infinitely large of order 1/4:

$$\frac{1}{\sqrt[4]{(1-x^4)}} : \frac{1}{\sqrt[4]{(1-x)}} = \frac{1}{\sqrt[4]{(1+x+x^2+x^3)}} \to \frac{1}{\sqrt[4]{4}} \quad \text{for} \quad x \to 1.$$

Hence the integral converges.

(2) $\int_0^{\pi/2} (\tan x)^p\,dx \qquad (p \neq 0)$.

If $p > 0$, then the singular point is $\pi/2$; when $p < 0$ the singular point is 0. In both cases the expression under the integral is infinitely large of order $|p|$. Thus there is convergence when $|p| < 1$ and divergence when $|p| > 1$.

(3) $\int_0^1 x^{a-1}(1-x)^{b-1}dx$.

When $a < 1$, the singular point is 0; when $b < 1$ the singular point is 1. We separate the integral given into two integrals, for instance, thus: $\int_0^1 = \int_0^{1/2} + \int_{1/2}^1$.

Because as $x \to 0$ the functions under the integrals become infinitely large (if $a < 1$) of order $1 - a$, the first integral exists only with the condition $1 - a < 1$, i.e. $a > 0$; analogously, the second exists when $b > 0$. Thus, the given integral converges if and only if at the same time $a > 0$ and $b > 0$.

(4) $\int_0^\infty x^{p-1} e^{-x}\, dx$.

The singular points are ∞ and 0 (when $p < 1$). $\int_0^1$ exists only when $p > 0$ (it is infinitely large of order $1 - p$ with respect to $1/x$). $\int_1^\infty$ exists whatever the value of p, because by taking $\lambda > 1$, we have

$$\frac{x^{p-1} e^{-x}}{1/x^\lambda} = \frac{x^{\lambda+p-1}}{e^x} \to 0 \quad \text{for} \quad x \to \infty;$$

$\int_0^\infty$ exists when $p > 0$.

Remark. In the first volume (p. 340) we mentioned the non-elementary function $\operatorname{li} y = \int (\log y)^{-1} dy$ ("the integral logarithm"). In order to fix the arbitrary constant contained in this integral, we set

$$\operatorname{li} y = \int_0^y \frac{dt}{\log t}.$$

When $y < 1$ this integral exists as a proper integral. However, when $y > 1$ the integral becomes meaningless, even as an improper integral, because when $t = 1$ the function under the integral is infinitely large to the first order (with respect to $1/(t-1)$). In this case the above integral can be regarded as the limit

$$\lim_{\varepsilon \to 0} \left\{ \int_0^{1-\varepsilon} + \int_{1+\varepsilon}^y \right\} \frac{dt}{\log t}.$$

We will not stop to prove the existence of such a limit.

In the case of an alternating function, by applying the Bolzano–Cauchy test, we have the following general condition for convergence:

For the convergence of the improper integral $\int_a^b f(x)\, dx$, *where* b *is a singular point, it is necessary and sufficient that to every number* $\varepsilon > 0$ *there corresponds a number* $\delta > 0$ *such that when* $0 < \eta < \delta$ *and* $0 < \eta' < \delta$ *the inequality*

$$\left| \int_{b-\eta}^{b-\eta'} f(x)\, dx \right| < \varepsilon$$

is satisfied.

Hence, as above, it follows that:

If the integral $\int_a^b |f(x)|\,dx$ *converges, then certainly the integral* $\int_a^b f(x)\,dx$ *converges.*

The converse, generally speaking, is false. Therefore here again we particularly distinguish the case when both the integrals $\int_a^b f(x)\,dx$ and $\int_a^b |f(x)|\,dx$ converge; then the first integral is said to be *absolutely convergent*, and the function $f(x)$ to be *absolutely integrable* over the interval $[a, b]$.

§ 3. Transformation and evaluation of improper integrals

291. Integration by parts in the case of improper integrals. Let the functions $u = u(x)$ and $v = v(x)$ be defined and continuous together with their first derivatives at all points of the interval $[a, b]$, excluding the point b (which may also be equal to $+\infty$). Then the equation

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

holds, if by the double substitution we understand the difference

$$\lim_{x \to b} u(x)v(x) - u(a)v(a).$$

Here it is presupposed that of the three expressions contained in the equation (the two integrals and the double substitution) two have finite values: this implies the finiteness of the third.

In fact, taking $a < x_0 < b$, we write down the usual formula for integration by parts in the interval $[a, x_0]$, where all the integrals are proper:

$$\int_a^{x_0} u\,dv = [u(x_0)v(x_0) - u(a)v(a)] - \int_a^{x_0} v\,du.$$

Now let x_0 tend to b in this equation. By our hypothesis, two of the expressions contained in it have finite limits as $x_0 \to b$†. Hence, the

† This is true also in the case when any of the integrals $\int_a^b$ are proper [Sec. 183, *(11)*].

third expression also has a finite limit, and the equation proved is justified by means of the passage to the limit.

Example.

$$\int_0^{\pi/2} \log \sin x \, dx = x \log \sin x \Big|_0^{\pi/2} - \int_0^{\pi/2} x \frac{\cos x}{\sin x} dx = -\int_0^{\pi/2} \frac{x}{\tan x} dx$$

—the integration by parts here succeeds in reducing the improper integral to a proper integral and thus proving the existence of the improper integral.

292. Change of variables in improper integrals. Let the function $f(x)$ be defined and continuous in the finite or infinite interval $[a, b)$, and, hence, integrable in the proper sense at each of its points, not including the point b, which may be $+\infty$; this point, by hypothesis, is the unique singular point for the function $f(x)$.

Now we will consider the monotonically increasing function $x = \varphi(t)$, continuous together with its derivative $\varphi'(t)$ in the interval $[\alpha, \beta)$, where β may be $+\infty$, and we suppose that $\varphi(\alpha) = a$ and $\varphi(\beta) = b$. The last equation must be understood in the sense that $\lim_{t \to \beta} \varphi(t) = b$.

With these conditions the equation

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)\,dt \tag{1}$$

holds, with the hypothesis that one of these integrals converges (the convergence of the other follows from this). The second integral will either be proper or improper—with a unique singular point.

By the theorem on the inverse function [Sec. 71] it is clear that t can also be considered as a monotonically increasing and continuous function of x in $[a, b)$: $t = \theta(x)$, where $\lim_{x \to b} \theta(x) = \beta$.

Now let x_0 and t_0 be arbitrary, but corresponding respectively to values of x and t in the intervals (a, b) and (α, β). Then by means of a substitution of the variable in the proper integral we will have

$$\int_a^{x_0} f(x)\,dx = \int_\alpha^{t_0} f(\varphi(t))\varphi'(t)\,dt.$$

If, say, the second of the integrals (1) converges, then we make x_0 approach b in an arbitrary way; here $t_0 = \theta(x_0)$ tends to β, and

we establish formula (1) simultaneously with the proof of the convergence of the integral on the left.

Our reasoning applies equally in the case when $\varphi(t)$ is monotonically decreasing, where $\alpha > \beta$†. In the same way we deal with all the other possible ways in which the singular points can be distributed. In the arrangement of the limits in the transformed integral one must always understand that *to the lower limit α must correspond the lower limit a, and to the upper limit β must correspond the upper limit b, independently of whether $\alpha < \beta$ or $\alpha > \beta$.*

Examples. (1) To calculate the integral

$$\int_a^b \frac{dx}{\sqrt{[(x-a)(b-x)]}}$$

by the substitution

$$x = a\cos^2\varphi + b\sin^2\varphi.$$

Here $\alpha = 0$, $\beta = \pi/2$, and the integral we are seeking is reduced to the proper integral

$$2\int_0^{\pi/2} d\varphi = \pi.$$

(2) In order to establish the convergence of the integral $\int_0^\infty \sin x^2\,dx$ we make a change of variable in it: $x = \sqrt{t}$, $dx = dt/2\sqrt{t}$, $a = \alpha = 0$, $b = \beta = \infty$. We obtain the well-known convergent [Sec. 287] integral $\frac{1}{2}\int_0^\infty \frac{\sin t}{\sqrt{t}}\,dt$, hence the given integral also converges. It is interesting to note that the function under the integral here does not tend to any limit as $x \to \infty$, but oscillates between -1 and 1.

We can solve the question of the convergence of the integral $\int_0^\infty \cos x^2\,dx$ analogously.

(3) The evaluation of the integral

$$I = \int_0^\infty \frac{dx}{1+x^4}$$

is made very simple by the application of appropriate substitutions.

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† Also in the case of improper integrals, when $\alpha > \beta$, we obtain

$$\int_\alpha^\beta = -\int_\beta^\alpha.$$

First of all, the integral

$$\int_0^\infty \frac{x^2\,dx}{1+x^4}$$

is reduced to it by means of the substitution $x = 1/t$ ($a = 0$, $b = \infty$, $\alpha = \infty$, $\beta = 0$), so that we can write

$$I = \frac{1}{2}\int_0^\infty \frac{(1+x^2)\,dx}{1+x^4} = \frac{1}{2}\int_0^\infty \frac{(1+1/x^2)\,dx}{x^2+1/x^2}.$$

If now we use the substitution $x - 1/x = z$ ($a = 0$, $b = \infty$, $\alpha = -\infty$, $\beta = \infty$), then we obtain at once

$$I = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dz}{z^2+2} = \frac{1}{2\sqrt{2}} \arctan \frac{z}{\sqrt{2}}\Bigg|_{-\infty}^{+\infty} = \frac{\pi}{2\sqrt{2}}.$$

293. The evaluation of integrals by artificial methods. We will calculate several important improper integrals by the application of artificial methods; the corresponding primitive functions here cannot be expressed in finite form, and it is impossible to make use of them. All these integrals were already known to mathematicians in the eighteenth century (in particular, to Euler), but without rigorous proof.

(1). We proceed to evaluate the integral

$$J = \int_0^{\pi/2} \log \sin x\,dx,$$

whose existence we have already ascertained [Sec. 291].

This evaluation is based on the use of a change of variable. Letting $x = 2t$, we have:

$$J = 2\int_0^{\pi/4} \log\sin 2t\,dt = \frac{\pi}{2}\log 2 + 2\int_0^{\pi/4} \log\sin t\,dt + 2\int_0^{\pi/4} \log\cos t\,dt.$$

By substituting in the last integral $t = \pi/2 - u$, we reduce it to the form $2\int_{\pi/4}^{\pi/2} \log\sin u\,du$, so that, finally, to determine J we obtain the equation

$$J = \frac{\pi}{2}\log 2 + 2J, \qquad \text{whence} \qquad J = -\frac{\pi}{2}\log 2.$$

(2). We turn to the evaluation of the integral

$$K = \int_0^\infty e^{-x^2}\,dx,$$

which occurs in the theory of probability†. For this purpose we first prove some inequalities.

By the usual methods of differential calculus it is not difficult to establish that the function $(1+t)e^{-t}$ attains its greatest value 1 when $t=0$. Hence, for $t \neq 0$ it will be true that

$$(1+t)e^{-t} < 1.$$

Here letting $t = \pm x^2$, we obtain

$$(1-x^2)e^{x^2} < 1 \quad \text{and} \quad (1+x^2)e^{-x^2} < 1,$$

whence

$$1 - x^2 < e^{-x^2} < \frac{1}{1+x^2}.$$

Limiting the variation in x in the first of these inequalities to the interval (0, 1) (so that $1-x^2 > 0$), and in the second assuming that x is some $x > 0$, we raise all these expressions to a power with any natural exponent n; this gives us‡

$$\underset{(0<x<1)}{(1-x^2)^n < e^{-nx^2}} \quad \text{and} \quad \underset{(x>0)}{e^{-nx^2} < \frac{1}{(1+x^2)^n}}.$$

Integrating the first inequality in the interval from 0 to 1, and the second from 0 to ∞, we obtain

$$\int_0^1 (1-x^2)^n\,dx < \int_0^1 e^{-nx^2}\,dx < \int_0^\infty e^{-nx^2}\,dx < \int_0^\infty \frac{dx}{(1+x^2)^n}.$$

But

$$\int_0^\infty e^{-nx^2}\,dx = \frac{1}{\sqrt{n}}\,K \qquad \text{(substituting } u = \sqrt{n}\cdot x\text{)},$$

$$\int_0^1 (1-x^2)^n\,dx = \int_0^{\pi/2} \sin^{2n+1} t\,dt = \frac{2n!!}{(2n+1)!!} \quad \text{(substituting } x = \cos t\text{)}$$

and finally

$$\int_0^\infty \frac{dx}{(1+x^2)^n} = \int_0^{\pi/2} \sin^{2n-2} t\,dt = \frac{(2n-3)!!}{(2n-2)!!}\,\frac{\pi}{2} \quad \text{(substituting } x = \cot t\text{)}.$$

---

† The existence of this integral is in no doubt, because (when $x > 1$) $e^{-x^2} < e^{-x}$, and the integral for e^{-x} can be evaluated immediately.

‡ It is permissible to raise an inequality with positive terms to a natural power termwise.

We have used here the well-known expression for $\int_0^{\pi/2} \sin^m x\,dx$ [Sec. 187, (5)]. Thus the desired value of K can be placed between the following two expressions:

$$\sqrt{n}\,\frac{2n!!}{(2n+1)!!} < K < \sqrt{n}\,\frac{(2n-3)!!}{(2n-2)!!}\,\frac{\pi}{2},$$

so that, squaring and rearranging:

$$\frac{n}{2n+1}\left[\frac{2n!!}{(2n-1)!!}\right]^2 \frac{1}{2n+1} < K^2 < \frac{n}{2n-1}\left[\frac{(2n-3)!!}{(2n-2)!!}\right]^2 (2n-1)\left(\frac{\pi}{2}\right)^2.$$

From Wallis's formula [Sec. 188]:

$$\frac{\pi}{2} = \lim_{n\to\infty}\left[\frac{2n!!}{(2n-1)!!}\right]^2 \frac{1}{2n+1}$$

it is easy to see that both the outside expressions tend to precisely the same limit $\pi/4$ as $n \to \infty$, hence,

$$K^2 = \pi/4 \quad \text{and} \quad K = \sqrt{\pi}/2 \qquad (\text{because } K > 0).$$

(3). We will consider, finally, the integral

$$L = \int_0^{\infty} \frac{\sin x}{x}\,dx,$$

which we already know to be convergent [Sec. 287]. If the definition of this improper integral

$$L = \lim_{A\to\infty} \int_0^{A} \frac{\sin x}{x}\,dx$$

is interpreted "in the terminology of sequences" [Sec. 32], then, in particular, we can set

$$L = \lim_{n\to\infty} \int_0^{n\pi/2} \frac{\sin x}{x}\,dx.$$

On the other hand, instead of the limit of a sequence we can look for [Sec. 234] the sum of the infinite series

$$\int_0^{\pi/2} + \left\{\int_0^{\pi} - \int_0^{\pi/2}\right\} + \left\{\int_0^{3\pi/2} - \int_0^{\pi}\right\} + \ldots = \int_0^{\pi/2} + \int_{\pi/2}^{\pi} + \int_{\pi}^{3\pi/2} + \ldots,$$

so that finally

$$L = \sum_{\nu=0}^{\infty} \int_{\nu\pi/2}^{(\nu+1)\pi/2} \frac{\sin x}{x}\,dx.$$

Letting $\nu = 2\mu$ or $2\mu - 1$ and making use, respectively, of the substitutions $x = \mu\pi + t$ or $x = \mu\pi - t$, we will have:

$$\int_{2\mu\,\pi/2}^{(2\mu+1)\,\pi/2} = (-1)^{\mu} \int_{0}^{\pi/2} \frac{\sin t}{\mu\pi + t}\, dt$$

and

$$\int_{(2\mu-1)\,\pi/2}^{2\mu\,\pi/2} = (-1)^{\mu-1} \int_{0}^{\pi/2} \frac{\sin t}{\mu\pi - t}\, dt.$$

Hence

$$L = \int_{0}^{\pi/2} \frac{\sin t}{t}\, dt + \sum_{\mu=1}^{\infty} \int_{0}^{\pi/2} (-1)^{\mu} \left(\frac{1}{t+\mu\pi} + \frac{1}{t-\mu\pi} \right) \sin t\, dt.$$

Because the series

$$\sum_{\mu=1}^{\infty} (-1)^{\mu} \left(\frac{1}{t+\mu\pi} + \frac{1}{t-\mu\pi} \right)$$

converges uniformly in the interval $0 \leqslant t \leqslant \pi/2$, because it is exceeded by the convergent series $\frac{1}{\pi} \sum_{1}^{\infty} \frac{1}{\mu^2 - 1/4}$, then—by multiplying all its terms by the bounded function $\sin t$—we can integrate it termwise. This enables us to write the expression for L in the form:

$$L = \int_{0}^{\pi/2} \sin t \left[\frac{1}{t} + \sum_{1}^{\infty} (-1)^{\mu} \left(\frac{1}{t+\mu\pi} + \frac{1}{t-\mu\pi} \right) \right] dt.$$

As we will see afterwards [Sec. 406, *(3)*, Remark], the expression in the brackets is equal to $1/(\sin t)$, which is the "expansion by simple fractions" for this function. Applying this fact now without proof, for the desired integral we finally obtain

$$L = \int_{0}^{\pi/2} dt = \frac{\pi}{2}.$$

This elegant deduction is due to Lobatchevsky, who was the first to turn his attention to the lack of rigorousness of the methods by which this important integral had previously been evaluated.

In fact, Lobatchevsky obtained by his method the more general formula:

$$\int_{0}^{\infty} f(x) \frac{\sin x}{x}\, dx = \int_{0}^{\pi/2} f(x)\, dx,$$

under the hypothesis that $f(x+\pi) = f(x)$ and $f(\pi - x) = f(x)$; in the case considered $f(x) = 1$.

Remark. The result we have found enables us to deduce that

$$L(a) = \int_0^\infty \frac{\sin at}{t}\,dt = \begin{cases} \pi/2 & \text{when} \quad a > 0, \\ 0 & \text{when} \quad a = 0, \\ -\pi/2 & \text{when} \quad a < 0. \end{cases}$$

When $a > 0$ this integral reduces to L by the simple substitution $x = at$; a change in the sign of a implies a change in the sign of the integral, and when $a = 0$ the integral is clearly equal to zero.

This simple remark, due to Fourier, made a strong impression on his contemporaries: the function of a, expressed differently for different values of this variable, at the same time admits a unique "analytic expression". [Cf. Sec. 18, *(3)* and below, Secs. 422 and 423.]

CHAPTER 18

INTEGRALS DEPENDING ON A PARAMETER

§ 1. Elementary theory

294. Statement of the problem. We will consider the function $f(x, y)$ of two variables, defined for all values of x in some—finite or infinite—interval $[a, b]$, and all values of y in the set $\mathcal{Y} = \{y\}$. For every constant value of y in $\mathcal{Y}$ let the function $f(x, y)$ be integrable in the interval $[a, b]$, in the proper or improper sense. Then the integral

$$I(y) = \int_a^b f(x, y)\,dx \tag{1}$$

will clearly be a function of the auxiliary variable or the parameter y.

While discussing in Sec. 269 the sequence of functions $\{f_n(x)\}$, we considered the integrals

$$I_n = \int_a^b f_n(x)\,dx,$$

which are a special case of the integrals (1): the part of the parameter here is played by the natural index n.

In connection with the function $I(y)$ a number of questions naturally arise—about the existence and expression of its limit by means of a definite passage to a limit, in particular, about its continuity with respect to y, about its differentiability and the expression for its derivative, and, finally, about its integral. In the present chapter we consider all these questions.

The study of the properties of the function expressed by the integral (1), depending on a parameter, is of independent interest (in this connection see, for instance, § 4). But, in addition to this,

these properties, as the reader can see, have widespread applications, in particular to the question of the evaluation of improper integrals.

295. Uniform approach to a limit function. An important part in the forthcoming investigations will be played by the concept of uniformity. Let the function $f(x, y)$ be defined, in the general case, in the two-dimensional set $\mathfrak{M} = \mathfrak{X} \times \mathfrak{Y}$, where $\mathfrak{X}$ and $\mathfrak{Y}$ denote the sets of values assumed separately by the variables x and y, where $\mathfrak{Y}$ has as its limit point, say, the finite number y_0.

If: $\{1\}$ *as* $y \to y_0$ *there exists a finite limit function*

$$\lim_{y \to y_0} f(x, y) = \varphi(x) \qquad (x \text{ in } \mathfrak{X}) \tag{2}$$

for the function $f(x, y)$, *and* $\{2\}$ *for any number* $\varepsilon > 0$ *we can find a number* $\delta > 0$ *not depending on* x *such that*

$$\text{when } |y - y_0| < \delta, \text{ then } |f(x, y) - \varphi(x)| < \varepsilon \tag{3}$$

at once for all x *in* $\mathfrak{X}$, *then we say that the function* $f(x, y)$ *tends to the limit function* $\varphi(x)$ *uniformly with respect to* x *in the domain* $\mathfrak{X}$.

It is not difficult to rephrase this definition for the case when y_0 is not a finite number, but for instance is $+\infty$: here the inequality of the form $|y - y_0| < \delta$ is merely replaced by an inequality of the form $y_0 > \Delta$. In Chapter 16 [Sec. 264] we discussed a special case of such a uniform approximation to the limit function, viz. the function $f_n(x)$ which contains as a parameter the natural index n.

In Sec. 265, in discussing a sequence of functions, we established that for uniform convergence the "uniform fulfilment", so to speak, of the principle of convergence is necessary and sufficient. We can do the same thing in the general case. In fact, if we limit ourselves to the hypothesis that y_0 is finite:

(1) In order that as $y \to y_0$ *the function* $f(x, y)$ *should have a limit function and should tend to it uniformly with respect to* x *in the domain* $\mathfrak{X}$, *it is necessary and sufficient that for every* $\varepsilon > 0$ *there should exist a number* $\delta > 0$ *not depending on* x *such that the inequality*

$$|f(x, y') - f(x, y)| < \varepsilon \tag{4}$$

is satisfied simultaneously for all x *in* $\mathfrak{X}$, *provided only*

$$|y - y_0| < \delta, \quad |y' - y_0| < \delta \qquad (y, y' \text{ in } \mathfrak{Y}). \tag{5}$$

[In the case when $y_0 = +\infty$, instead of the last inequalities we have the inequalities $y > \Delta$, $y' > \Delta$.]

Necessity. Let uniform convergence hold. Replacing ε by $\varepsilon/2$ in the definition and choosing δ correspondingly, let us now take two values y and y' from $\mathcal{Y}$ such that the conditions (5) are satisfied. Then, whatever the value of x, we will have

$$|f(x, y') - \varphi(x)| < \varepsilon/2 \quad \text{and} \quad |\varphi(x) - f(x, y)| < \varepsilon/2,$$

hence follows (4).

Sufficiency. If the given condition is satisfied, then first of all it is clear that the limit function (2) exists. Then taking limits in the inequality (4) as $y' \to y_0$ (where y is fixed so that $|y - y_0| < \delta$), we obtain:

$$|\varphi(x) - f(x, y)| \leqslant \varepsilon.$$

This establishes the fact that the function $f(x, y)$ tends uniformly to the limit function $\varphi(x)$.

Now let the set $\mathcal{X}$ be a finite interval $[a, b]$. By Theorem 1, Sec. 266, if the sequence $\{f_n(x)\}$ of continuous functions converges uniformly to the limit function, then this latter will necessarily also be continuous; this assertion carries over to the general case:

(2) If for any y in $\mathcal{Y}$ the function $f(x, y)$ is continuous with respect to x in the interval $\mathcal{X} = [a, b]$ and as $y \to y_0$ tends uniformly to the limit function $\varphi(x)$, then this latter function will also be continuous.

In fact, we need only take any sequence $\{y_n\}$ in $\mathcal{Y}$ which has the limit y_0 in order that the corresponding sequence of functions $\{f(x, y_n)\}$ should converge uniformly to $\varphi(x)$: for instance, for finite y_0 the inequality

$$|f(x, y_n) - \varphi(x)| < \varepsilon$$

will be satisfied for all x in $\mathcal{X}$, provided only that n is large enough that $|y_n - y_0| < \delta$ [see (3)]. It only remains to appeal to the theorem mentioned.

It is also easy to generalize Dini's theorem [Sec. 267, Theorem 2]; on this occasion we presuppose that all $y < y_0$.

(3) Now for any y in $\mathcal{Y}$ let the function $f(x, y)$ be continuous with respect to x in the interval $\mathcal{X} = [a, b]$ and as y increases, let it be monotonically increasing and let it converge to the continuous

limit function $\varphi(x)$. Then this convergence will necessarily be uniform with respect to x in the interval $\mathfrak{X}$.

To prove this we choose from $\mathfrak{Y}$ a monotonically increasing sequence $\{y_n\}$ of values of y, tending to y_0, and we consider the corresponding sequence of functions $\{f(x, y_n)\}$, which, clearly, also increases monotonically with n.

Dini's theorem enables us to assert that this sequence converges to $\varphi(x)$ uniformly with respect to x in the interval $\mathfrak{X}$. Hence, for a given $\varepsilon > 0$ we can find a number n_0 such that the inequality

$$|\varphi(x) - f(x, y_{n_0})| < \varepsilon$$

is satisfied at once for all x in $\mathfrak{X}$. In view of the fact that $f(y)$ increases monotonically together with y, then certainly the inequality

$$|\varphi(x) - f(x, y)| < \varepsilon$$

is also satisfied, provided only that $y > y_{n_0}$; thus we have proved our assertion.

Although the special test established for uniform approximation seems very restrictive, it is often useful in eliminating the necessity for ascertaining by other means the presence of uniform approximation.

296. Taking limits under the integral sign. Now we turn to the consideration of the integral (1), depending on the parameter y, limiting ourselves in the first case to the finite interval $[a, b]$ and a continuous function.

Presupposing that the domain $\mathfrak{Y}$ of variation of the parameter y has the limit point y_0, we raise the question of the limit of the function (1) as $y \to y_0$.

THEOREM 1. *If the function $f(x, y)$ for a constant y is continuous with respect to x in $[a, b]$ and as $y \to y_0$ tends to the limit function* (2) *uniformly with respect to x, then the equation*

$$\lim_{y \to y_0} I(y) = \lim_{y \to y_0} \int_a^b f(x, y)\, dx = \int_a^b \varphi(x)\, dx \tag{6}$$

holds.

Proof†. The continuity of the limit function $\varphi(x)$ is already known [Sec. 295, *(2)*]. Given an arbitrary number $\varepsilon > 0$, we can

† In order to be specific, we presuppose that y_0 is finite.

find a number $\delta > 0$ such that (3) holds. Then when $|y - y_0| < \delta$ we will have

$$\left| \int_a^b f(x, y)\, dx - \int_a^b \varphi(x)\, dx \right| = \left| \int_a^b [f(x, y) - \varphi(x)]\, dx \right|$$

$$\leqslant \int_a^b |f(x, y) - \varphi(x)|\, dx < \varepsilon(b - a),$$

which proves (6).

Formula (6) can be rewritten in the form:

$$\lim_{y \to y_0} \int_a^b f(x, y)\, dx = \int_a^b \lim_{y \to y_0} f(x, y)\, dx.$$

When this occurs, we say that *it is permissible to take limits with respect to the parameter "under the integral sign"*.

Presupposing that all $y < y_0$, we have on the basis of Dini's generalized theorem [Sec. 295, *(3)*]:

COROLLARY. *If the function $f(x, y)$ for constant y is continuous with respect to x in $[a, b]$ and as y increases tends to a continuous limit function, which is monotonically increasing, then formula* (6) *is valid.*

THEOREM 2. *If the function $f(x, y)$ is defined and continuous as a function of two variables, in the rectangle $[a, b; c, d]$, then the integral* (1) *will be a continuous function of the parameter y in the interval $[c, d]$.*

Proof. In view of the uniform continuity of the function $f(x, y)$ [Sec. 137], for an arbitrary $\varepsilon > 0$ we can find a $\delta > 0$ such that from the inequality

$$|x'' - x'| < \delta, \quad |y'' - y'| < \delta$$

follows the inequality

$$|f(x'', y'') - f(x', y')| < \varepsilon.$$

We set, in particular, $x' = x'' = x$, $y' = y_0$, $y'' = y$; then when $|y - y_0| < \delta$, whatever may be the value of x, we will have

$$|f(x, y) - f(x, y_0)| < \varepsilon.$$

Thus, as y tends to any particular value y_0, the function $f(x, y)$ tends to the function $f(x, y_0)$ uniformly with respect to x. In this case, by Theorem 1,

$$\lim_{y \to y_0} \int_a^b f(x, y)\, dx = \int_a^b f(x, y_0)\, dx$$

or

$$\lim_{y \to y_0} I(y) = I(y_0),$$

which proves our assertion.

297. Differentiation under the integral sign. In the study of the properties of the function (1) which is given as an integral containing the parameter y, the question of the derivative of this function with respect to the parameter is of great importance.

With the well-known hypotheses for the evaluation of the derivative $I'(y)$, the formula

$$I'(y) = \int_a^b f'_y(x, y)\, dx \tag{7}$$

can be applied, or—if we use the more expressive notation of Cauchy—

$$D_y \int_a^b f(x, y)\, dx = \int_a^b D_y f(x, y)\, dx.$$

If the symbols for the derivative with respect to y and the integral with respect to x can be commuted like this, then we say that the function (1) *can be differentiated with respect to the parameter "under the integral sign"*.

A similar method was first communicated by Leibniz in 1697 in a letter to Johann Bernoulli, by whom it was received enthusiastically. This method of evaluating the derivative by the formula given is called "Leibniz's rule".

The following theorem establishes the simple conditions sufficient for the rule to be applicable.

THEOREM 3. *Let the function $f(x, y)$, defined in the rectangle $[a, b; c, d]$, be continuous with respect to x in $[a, b]$ for any constant y in $[c, d]$. We presuppose, further, that in the whole domain there*

exists a partial derivative $f'_y(x, y)$, continuous as a function of two variables†. *Then formula* (7) *holds for any y in $[c, d]$.*

The continuity of the function $f(x, y)$ with respect to x guarantees the existence of the integral (1).

Fixing any value of $y = y_0$, we add to it the increment $\Delta y = k$. Then

$$I(y_0) = \int_a^b f(x, y_0)\,dx, \qquad I(y_0 + k) = \int_a^b f(x, y_0 + k)\,dx,$$

so that

$$\frac{I(y_0 + k) - I(y_0)}{k} = \int_a^b \frac{f(x, y_0 + k) - f(x, y_0)}{k}\,dx. \tag{8}$$

The integral on the right depends on the parameter k. We have to prove that as $k \to 0$ we may here take limits under the integral sign. In this way the existence of the derivative

$$I'(y_0) = \lim_{k \to 0} \frac{I(y_0 + k) - I(y_0)}{k}$$

will also be established, and the existence of the required equation

$$I'(y_0) = \lim_{k \to 0} \int_a^b \frac{f(x, y_0 + k) - f(x, y_0)}{k}\,dx$$

$$= \int_a^b \lim_{k \to 0} \frac{f(x, y_0 + k) - f(x, y_0)}{k}\,dx = \int_a^b f'_y(x, y_0)\,dx.$$

For this purpose we first write down by means of Lagrange's formula

$$\frac{f(x, y_0 + k) - f(x, y_0)}{k} = f'_y(x, y_0 + \theta k) \qquad (0 < \theta < 1). \tag{9}$$

Making use of the uniform continuity of the function $f'_y(x, y)$, for an arbitrary $\varepsilon > 0$ we can find a $\delta > 0$ such that when

$$|x'' - x'| < \delta \quad \text{and} \quad |y'' - y'| < \delta$$

the inequality

$$|f'_y(x'', y'') - f'_y(x', y')| < \varepsilon$$

will be satisfied.

† From these conditions, in fact, follows the continuity of the function $f(x, y)$ with respect to both arguments, but we will not make use of it.

Here letting $x' = x'' = x$, $y' = y_0$, $y'' = y_0 + \theta k$ and assuming $|k| < \delta$, we obtain, taking (9) into account, that at once for all x

$$\left| \frac{f(x, y_0 + k) - f(x, y_0)}{k} - f'_y(x, y_0) \right| < \varepsilon$$

will hold. Hence it is clear that the function (9) under the integral tends, as $k \to 0$, to the limit function $f'_y(x, y_0)$ uniformly with respect to x. This, by Theorem 1, justifies taking limits under the integral sign (8).

298. Integration under the integral sign. Finally, we raise the question of the integral with respect to y of the function (1), say in the interval $[c, d]$.

We will be particularly interested in the case when this integral can be expressed by the formula:

$$\int_c^d I(y)\,dy = \int_c^d \left\{ \int_a^b f(x, y)\,dx \right\} dy = \int_a^b \left\{ \int_c^d f(x, y)\,dy \right\} dx,$$

which—without the brackets—is usually written in the form:

$$\int_c^d dy \int_a^b f(x, y)\,dx = \int_a^b dx \int_c^d f(x, y)\,dy. \tag{10}$$

When this happens we say that the function (1) *can be integrated with respect to the parameter y "under the integral sign"* (taken with respect to x).

The simplest conditions, sufficient for the equation in the two repeated integrals (10), are given by

THEOREM 4. *If the function $f(x, y)$ is continuous (with respect to both variables) in the rectangle $[a, b; c, d]$, then formula* (10) *holds.*

We will prove the more general equation

$$\int_c^\eta dy \int_a^b f(x, y)\,dx = \int_a^b dx \int_c^\eta f(x, y)\,dy, \tag{10a}$$

where $c \leqslant \eta \leqslant d$.

On the left and right sides of this we have two functions of the parameter η; we will evaluate their derivatives with respect to η.

The outside integral on the left has as the function under the integral (1), which is continuous with respect to y by Theorem 2.

Therefore its derivative with respect to the variable in the upper limit will be equal to the function under the integral evaluated when $y = \eta$, i.e. the integral

$$I(\eta) = \int_a^b f(x, \eta)dx.$$

On the right-hand side of (10a) is the integral

$$\int_a^b \varphi(x, \eta)dx, \quad \text{where} \quad \varphi(x, \eta) = \int_c^\eta f(x, y)dy;$$

the function $\varphi(x, \eta)$ satisfies the conditions of Theorem 3. In fact, $\varphi(x, \eta)$ is continuous with respect to x, by Theorem 2†. Then, the derivative

$$\varphi'_\eta(x, \eta) = f(x, \eta)$$

is continuous as a function of two variables. Therefore Leibniz's rule is applicable to the given integral:

$$D_\eta \int_a^b \varphi(x, \eta)dx = \int_a^b \varphi'_\eta(x, \eta)dx = \int_a^b f(x, \eta)dx = I(\eta).$$

Thus, the left and right sides of equation (10a), as functions of η, have equal derivatives, hence they can differ at most by a constant. But when $\eta = c$ both the given expressions clearly vanish; consequently they are identical for all values of η, and the equation (10a) is proved.

When $\eta = d$, in particular, we obtain from this equation (10).

Remark. This equation was first mentioned by Euler in 1769 (published by him in the following year). He only illustrated it by an example, "because its basis is completely clear from the nature of the differentials and integrals".

299. The case when the limits of the integral also depend on the parameter. We turn to the consideration of a more complicated case, when not only the expression under the integral contains the parameter, but also the limits of the integral depend on it. In this case the integral has the form

$$I(y) = \int_{\alpha(y)}^{\beta(y)} f(x, y)dx. \tag{11}$$

We will limit ourselves to the problem of the continuity and the differentiability with respect to the parameter of such an integral.

† Here the part of the parameter is played by x.

THEOREM 5. *Let the function $f(x, y)$ be defined and continuous in the rectangle $[a, b; c, d]$, and the curves*

$$x = \alpha(y), \quad x = \beta(y) \quad (c \leqslant y \leqslant d)$$

be continuous and not falling outside its limits. Then the integral (11) *is a continuous function of y in $[c, d]$.*

If y_0 is any particular value of y, then the integral (11) can be written in the form

$$I(y) = \int_{\alpha(y_0)}^{\beta(y_0)} f(x, y)dx + \int_{\beta(y_0)}^{\beta(y)} f(x, y)dx - \int_{\alpha(y_0)}^{\alpha(y)} f(x, y)dx. \tag{12}$$

The first integral, in which the limits are constants, tends as $y \to y_0$, to

$$I(y_0) = \int_{\alpha(y_0)}^{\beta(y_0)} f(x, y_0)dx,$$

by Theorem 2. The remaining two integrals may be approximated:

$$\left| \int_{\beta(y_0)}^{\beta(y)} f(x, y)dx \right| \leqslant M|\beta(y) - \beta(y_0)|,$$

$$\left| \int_{\alpha(y_0)}^{\alpha(y)} f(x, y)dx \right| \leqslant M|\alpha(y) - \alpha(y_0)|$$

where $M = \max |f(x, y)|$, and, by virtue of the continuity of the functions $\alpha(y)$ and $\beta(y)$—as $y \to y_0$ they tend to zero.

Thus, finally

$$\lim_{y \to y_0} I(y) = I(y_0),$$

which proves the theorem.

THEOREM 6. *If, in addition to the previous hypotheses, the function $f(x, y)$ has a continuous derivative $f'_y(x, y)$ in the rectangle $[a, b; c, d]$, and also there exist derivatives $\alpha'(y)$ and $\beta'(y)$, then the integral* (11) *has a derivative with respect to its parameter, which can be expressed by the formula*

$$I'(y) = \int_{\alpha(y)}^{\beta(y)} f'_y(x, y)dx + \beta'(y) \cdot f(\beta(y), y) - \alpha'(y) \cdot f(\alpha(y), y). \tag{13}$$

Here we will start from equation (12). When $y = y_0$ the first integral has a derivative, represented by the integral of the derivative

$$\int_{\alpha(y_0)}^{\beta(y_0)} f'_y(x, y_0)dx$$

—by Theorem 3. For the second integral (the value of which when $y = y_0$ is zero) we have, by the mean value theorem:

$$\frac{1}{y-y_0}\int_{\beta(y_0)}^{\beta(y)} f(x, y)dx = \frac{\beta(y)-\beta(y_0)}{y-y_0} f(\bar{x}, y),$$

where $\bar{x}$ is contained between $\beta(y_0)$ and $\beta(y)$. Hence, when $y = y_0$ the derivative of the second integral, which coincides with the limit of the preceding expression as $y \to y_0$, will be

$$\beta'(y_0)\cdot f(\beta(y_0), y_0).$$

Analogously, for the derivative of the third integral when $y = y_0$ we obtain

$$-\alpha'(y_0)\cdot f(\alpha(y_0), y_0).$$

Combining all these results, we ascertain that the derivative $I'(y_0)$ exists and is defined by the given formula.

Remark. The conclusions of both theorems remain valid also under the hypothesis that the function $f(x, y)$ is given (and possesses the given properties) only in the domain lying between the curves

$$x = \alpha(y) \quad \text{and} \quad x = \beta(y).$$

The possibility of considering the function outside this domain was used in order to simplify the reasoning.

It is instructive to consider briefly the results established from the following point of view: the integral $I(y)$ is obtained from the integral

$$I(y, u, v) = \int_u^v f(x, y)\, dx,$$

depending on three parameters y, u, v, where $u = \alpha(y)$, $v = \beta(y)$. The question is solved by applying the general theorems on the continuity and differentiability of a compound function. In particular, formula (13) can be written in the classical form:

$$\frac{dI}{dy} = \frac{\partial I}{\partial y} + \frac{\partial I}{\partial u}\alpha'(y) + \frac{\partial I}{\partial v}\beta'(y).$$

300. Examples. *(1)* To evaluate the integral

$$I = \int_0^1 \frac{x^b - x^a}{\log x}\, dx \qquad (0 < a \leqslant b).$$

Representing the expression under the integral, in turn, in the form of the integral

$$\frac{x^b - x^a}{\log x} = \int_a^b x^y\, dy$$

and changing the order—by Theorem 4—in the repeated integral

$$I = \int_0^1 dx \int_a^b x^y \, dy$$

integrated with respect to x and y, we at once obtain the result

$$I = \int_a^b dy \int_0^1 x^y \, dx = \int_a^b \frac{dy}{y+1} = \log \frac{b+1}{a+1}.$$

(2) Alternatively, this result can also be obtained as follows; we regard the integral I—for fixed a—as a function of the variable parameter $b (b \geqslant a)$. By Theorem 3:

$$\frac{dI}{db} = \int_0^1 x^b \, dx = \frac{1}{b+1}.$$

Hence,

$$I = \log(b+1) + C.$$

But when $b = a$ the integral I is clearly zero, so that $C = -\log(a+1)$. By substitution we arrive at the same result as above.

(3) In the cases considered it was immediately clear that the conditions of Theorem 3 or 4 were satisfied. If these conditions are violated we can conclude that the theorem is not true. We give an example of this.

Let $f(x, y) = \dfrac{y^2 - x^2}{(x^2+y^2)^2}$ in the rectangle $[0, 1; 0, 1]$. Then

$$\int_0^1 f(x, y) \, dx = \frac{x}{x^2+y^2} \Big|_{x=0}^{x=1} = \frac{1}{1+y^2} \quad (y > 0),$$

$$\int_0^1 dy \int_0^1 f(x, y) \, dx = \arctan y \Big|_0^1 = \frac{\pi}{4},$$

and

$$\int_0^1 dx \int_0^1 f(x, y) \, dy = -\frac{\pi}{4}.$$

We note that at the point $(0, 0)$ the function $f(x, y)$ has a discontinuity.

§ 2. Uniform convergence of integrals

301. The definition of uniform convergence of integrals. In extending to the case of improper integrals the theory of integrals depending on a parameter that we have presented, a special part is played by the concept of the uniform convergence of integrals, which we will first explain.

We presuppose that the function $f(x, y)$ is given for all values $x \geqslant a$ and all values of y in some domain $\mathcal{Y}$. Further, let the integral

$$I(y) = \int_a^\infty f(x, y)\, dx \tag{1}$$

exist for every y in this domain.

By the same definition of an improper integral with an infinite limit [Sec. 282]:

$$\int_a^\infty f(x, y)\, dx = \lim_{A \to \infty} \int_a^A f(x, y)\, dx.$$

Thus, the integral

$$F(A, y) = \int_a^A f(x, y)\, dx, \tag{2}$$

being a function of A and y, where $y = \text{const}$ and $A \to \infty$, has the limit $I(y)$. *If this integral tends uniformly to $I(y)$ with respect to y in the domain $\mathcal{Y}$, then the integral $I(y)$ is said to be uniformly convergent with respect to y for the given values of the parameter.*

This means that for any $\varepsilon > 0$ we can find a number $A_0 \geqslant a$ not depending on y, such that, provided only $A > A_0$, the inequality

$$\left| \int_a^\infty f(x, y)\, dx - \int_a^A f(x, y)\, dx \right| = \left| \int_A^\infty f(x, y)\, dx \right| < \varepsilon$$

will be satisfied simultaneously for all values of y in $\mathcal{Y}$.

If we compare the integral (1), which depends on the parameter y, with the infinite series of functions

$$\sum_{n=1}^{\infty} u_n(x),$$

where the parameter is x, and if we recall what was said in Sec. 284 about the analogy between improper integrals and infinite series, then a complete parallel is obvious between the definition given above and the definition of a uniformly convergent series in Sec. 264.

We note that Zaidel and Stokes, who first contrasted uniform and non-uniform convergence of functional series [Sec. 281], at the same time indicated a similar possibility for the integral $\int_a^\infty$, containing a parameter.

As an example we consider the integral

$$\int_0^\infty ye^{-xy}\,dx,$$

which converges for every fixed value $y \geqslant 0$.

We can immediately calculate the integral

$$\int_A^\infty ye^{-xy}\,dx.$$

When $y = 0$ it is equal to 0, for all A; if, however, $y > 0$, then by the substitution $xy = t$ we can easily find

$$\int_A^\infty ye^{-xy}\,dx = \int_{Ay}^\infty e^{-t}\,dt = e^{-Ay}.$$

When y is fixed, as $A \to \infty$ this expression clearly tends to 0, and for any $\varepsilon > 0$, the inequality

$$e^{-Ay} < \varepsilon \tag{3}$$

will be satisfied for all $A > A_0(y)$, where

$$A_0(y) = \frac{\log 1/\varepsilon}{y}$$

depends on y.

If the variation in y is limited to the interval $[c, d]$, where $c > 0$, then we can find a number A_0 not depending on y such that when $A > A_0$ the inequality (3) will be satisfied simultaneously for all y: it is sufficient to take $A_0(c)$ for A_0, because when $A > A_0$, then it will be true that

$$e^{-Ay} \leqslant e^{-Ac} < \varepsilon \qquad (c \leqslant y \leqslant d).$$

In other words, our integral converges absolutely with respect to y in the interval $[c, d]$.

The situation is different if the parameter y varies throughout the interval $[0, d]$ $(d > 0)$. But this time such an A_0 does not exist (at any rate, if $\varepsilon < 1$). This can be seen merely from the fact that however large we take A, the expression e^{-Ay} tends to 1 as $y \to 0$, so that for sufficiently small values of y it will be greater than any number $\varepsilon < 1$. The convergence of the integral for values of y in the interval $[0, d]$ will not be uniform with respect to y.

302. Conditions and sufficiency tests for uniform convergence. Using the general criterion for the uniform approach of a function to a limit [Sec. 295, *(1)*], we can formulate it in a form suitable for the case under consideration in the following way:

In order that the integral (1) *should converge uniformly with respect to y in the domain* $\mathcal{Y}$, *it is necessary and sufficient that for any given* $\varepsilon > 0$ *we should be able to find a number* A_0 *not depending on y, such that the inequality*

$$\left| \int_a^{A'} f(x, y)\,dx - \int_a^{A} f(x, y)\,dx \right| = \left| \int_A^{A'} f(x, y)\,dx \right| < \varepsilon$$

is satisfied at the same time for all y in $\mathcal{Y}$, *provided only that* $A' > A > A_0$.

Now we will establish some tests, by which in practice we can usually determine whether the integrals converge uniformly.

(1) We will presuppose the function $f(x, y)$ is continuous with respect to x for $x \geqslant a$. *If there exists a function* $\varphi(x,)$ *depending only on x and integrable in the infinite interval* $[a, \infty]$, *such that for all values of y in* $\mathcal{Y}$

$$|f(x, y)| \leqslant \varphi(x) \qquad (\textit{for } x \geqslant a),$$

then the integral (1) *converges uniformly with respect to y (in the given domain for its values).*

This follows immediately from the inequality

$$\left| \int_A^{A'} f(x, y)\,dx \right| \leqslant \int_A^{A'} |f(x, y)|\,dx \leqslant \int_A^{A'} \varphi(x)\,dx$$

by just using the criterion we have proved.

With the given conditions one sometimes says that the function $f(x, y)$ has an integrable majorant $\varphi(x)$, or that the integral (1) is majorized by the convergent integral

$$\int_a^{\infty} \varphi(x)\,dx,$$

which does not contain the parameter.

It is easy to see the resemblance between this test and the well-known test for the uniform convergence of a functional series, due to Weierstrass [Sec. 265].

Example. {1} The integral

$$\int_0^\infty \frac{\cos ax}{k^2+x^2}\,dx \qquad (k \neq 0)$$

converges uniformly with respect to the parameter a (in any domain of variation for it), because it is exceeded by the convergent integral

$$\int_0^\infty \frac{dx}{k^2+x^2},$$

not containing a.

(2) We consider the integrals

$$\int_a^\infty f(x,y)\cdot g(x)\,dx \qquad (4a), \qquad \int_a^\infty f(x)\cdot g(x,y)\,dx \qquad (4b)$$

of the products of two functions, of which only one contains the parameter y, varying throughout some domain $\mathcal{Y}$, and we formulate for them more subtle tests for uniform convergence, which are modelled on the tests in Sec. 287.

The functions f and g (if necessary—when $y = \text{const}$) we assume, as we did above, to be continuous functions of x, and g, moreover, to be monotonic and to have a continuous derivative with respect to x. Then

(I) *the integral* (4a) *converges uniformly with respect to* y *in* $\mathcal{Y}$, *if*

[1] *the integral*

$$\Phi(A,y) = \int_a^A f(x,y)\,dx$$

is uniformly bounded for all A *and all* y:

$$|\Phi(A,y)| \leqslant K \qquad (K = \text{const})$$

and [2] $g(x) \to 0$ *as* $x \to \infty$;

similarly

(II) *the integral* (4b) *converges uniformly with respect to* y, *if*

[1′] *the integral*

$$\int_a^\infty f(x)\,dx$$

converges, and

[2′] *the function* $g(x,y)$ *is uniformly bounded for all* x *and all* y:

$$|g(x,y)| \leqslant L \qquad (L = \text{const}).$$

The proof is essentially similar to the previous one, because—since the constants K and L do not depend on y—the evaluation of the integral

$$\int_A^{A'} f\cdot g\,dx$$

is obtained independently of y, which guarantees the uniform convergence.

Examples. {2} The integral

$$\int_0^\infty \frac{x \sin ax}{k^2 + x^2}\, dx$$

converges uniformly with respect to a, for $a \geqslant a_0 > 0$—by the test (I). In fact, setting $f(x, a) = \sin ax$, we have

$$\left| \int_0^A \sin ax\, dx \right| = (1 - \cos aA)/a \leqslant 2/a_0 = K;$$

on the other hand, the function

$$g(x) = x/(k^2 + x^2)$$

is monotonically decreasing (at least, for sufficiently large x), and tends to 0 as $x \to \infty$.

{3} The integral

$$\int_0^\infty e^{-ax} \frac{\sin x}{x}\, dx$$

converges uniformly with respect to a, for $a \geqslant 0$—by the test (II). Here $f(x) = (\sin x)/x$ is integrable from 0 to ∞, and the monotonically decreasing (in a broad sense) function $g(x, a) = e^{-ax}$ is bounded by.

303. The case of integrals with finite limits. Now we consider the function $f(x, y)$, defined for values of x in the finite interval $[a, b]$ and values of y in some domain $\mathcal{Y}$; when $y = \text{const.}$ let it be integrable with respect to x (in the proper sense or otherwise) from a to b. Then the integral

$$I(y) = \int_a^b f(x, y)\, dx, \tag{5}$$

be it proper or not, is the limit as $\eta \to 0$ of the integral

$$\varphi(\eta, y) = \int_a^{b-\eta} f(x, y)\, dx. \tag{6}$$

If as $\eta \to 0$ this integral tends to the limit $I(y)$ uniformly with respect to y for values of y in the domain $\mathcal{Y}$, then we say that the integral (5) *converges uniformly with respect to y in the given domain.*

This means that for any $\varepsilon > 0$ we can find a number $\delta > 0$ not depending on y, such that, provided only that $\eta \leqslant \delta$, the inequality

$$\left| \int_a^b f(x, y)\, dx - \int_a^{b-\eta} f(x, y)\, dx \right| = \left| \int_{b-\eta}^b f(x, y)\, dx \right| < \varepsilon$$

will be satisfied at the same time for all values of y in $\mathcal{Y}$.

It is not difficult to formulate here a condition necessary and sufficient for uniform convergence, which is similar to the sufficiency tests in Sec. 302 for the case under consideration. We leave this to the reader.

We considered the integral (5) from a to b as the limit of the integral (6) from a to $b-\eta$ and we were interested in the nature of the approximation of the last integral to its limit. Thus, a special part is played here by the point $x=b$ (as in Sec. 301 by the point $x=\infty$). It may become necessary (depending on circumstances which will be explained further) to consider in a similar manner other points in the interval. For instance, the same integral (5) can be considered as the limit as $\eta \to 0$ of the integral

$$\psi(\eta, y) = \int_{a+\eta}^{b} f(x, y)dx.$$

If as $\eta \to 0$ the latter approximates to its limit uniformly with respect to y, then we also say that the integral (5) is uniformly convergent. All that has been said above also carries over to this case.

If it is possible for there to be any doubt as to what sort of uniform convergence we are discussing, we say that *the integral converges uniformly* (with respect to y in a definite domain), respectively, *when* $x=\infty$, *when* $x=b$, *when* $x=a$, etc.

We note that, as a rule, the uniform convergence of the integral (5), say, when $x=b$, interests us in the cases when the point $x=b$ is singular for the integral (5) in the sense of Sec. 288—for some value of y. But the definition not only remains in force formally when the integral (5) is proper for all values, but, as we see, is really useful also in this case.

For instance, the integral

$$\int_0^1 \frac{y}{x^2+y^2}\,dx$$

for every value of y in the interval $[0, d]$, where $d>0$, will exist as a proper integral. However, for the given interval of variation of y its convergence will not be uniform when $x=0$. In fact, the inequality

$$\int_0^\eta \frac{y}{x^2+y^2} = \arctan\frac{\eta}{y} < \varepsilon,$$

if merely $\varepsilon < \pi/2$, cannot be satisfied at the same time for all values of $y > 0$: however small we take η, its left side, as $y \to 0$, tends to $\pi/2$ and, for sufficiently small values of y, will certainly be greater than ε.

§ 3. The use of the uniform convergence of integrals

304. Taking limits under the integral sign. Now we will turn to the study of the integral with the infinite limit

$$I(y) = \int_a^\infty f(x, y)dx, \tag{1}$$

which depends on the parameter y (varying in the domain $\mathcal{Y}$), and we will prove for it a series of theorems, resembling those in Secs. 296–298, established for integrals in the proper sense. The transition from one to the other will be realized by means of the same method which we followed in Secs. 266–270 for carrying over the properties of a finite sum of functions to the case of the sum of an infinite series of functions. The role played here by the uniform convergence of an integral of the form (1) is similar to that previously played by the uniform convergence of a series of functions. Thus, we will again be guided by the analogy between infinite series and improper integrals, which we have more than once emphasized [Secs. 284, 301].

We begin with the question of taking limits under an integral sign extending over an infinite interval.

Theorem 1, Sec. 296 does not extend to this case: even if, in the whole infinite interval, as $y \to y_0$ the function $f(x, y)$ tends uniformly to the limit function $\varphi(x)$, it may not be permissible to take limits under the integral sign.

We consider, by way of an example, the function ($n = 1, 2, 3, \ldots$):

$$\begin{cases} f_n(x) = \dfrac{n}{x^3} e^{-n/2x^2} & (x > 0) \\ f_n(0) = 0. \end{cases}$$

By the usual methods of differential calculus it is easy to establish that the greatest value this function attains is at $x = \sqrt{(n/3)}$, and it is equal to $3\sqrt{\{3/(ne^3)\}}$. Because as $n \to \infty$ this value tends to zero, it is clear from this that the function $f_n(x)$ as $n \to \infty$ tends

uniformly to $\varphi(x) = 0$ in the whole interval $[0, \infty)$. Nevertheless, the integral

$$\int_0^\infty f_n(x)dx = 1$$

as $n \to \infty$ certainly does not tend to zero.

Conditions sufficient to ensure that one may take limits are given in the following theorem:

THEOREM 1. *Let the function $f(x, y)$, defined for $x \geqslant a$ and for y in $\mathcal{Y}$, be: (1) continuous with respect to x and (2) as $y \to y_0$ tend to the limit function $\varphi(x)$ uniformly with respect to x in every finite interval $[a, A]$. If, moreover, (3) the integral* (1) *converges uniformly with respect to y in the domain $\mathcal{Y}$, then*

$$\lim_{y \to y_0} \int_a^\infty f(x, y)dx = \int_a^\infty \varphi(x)dx. \tag{2}$$

We note, first of all, that the function $\varphi(x)$ will also be continuous [Sec. 295, *(2)*]. By the condition of uniform convergence [Sec. 302] for any $\varepsilon > 0$ we can find an $A_0 > a$ such that

$$\left| \int_A^{A'} f(x, y)dx \right| < \varepsilon$$

for all y in $\mathcal{Y}$ at the same time, provided only that $A' > A > A_0$. Taking limits here as $y \to y_0$ under the integral sign [Sec. 296, Theorem 1], we obtain

$$\left| \int_A^{A'} \varphi(x)dx \right| < \varepsilon,$$

whence it follows [Sec. 286] that the function $\varphi(x)$ is integrable in the infinite interval $[0, \infty]$.

Further, for any $A > a$, we have:

$$\left| \int_a^\infty f(x, y)dx - \int_a^\infty \varphi(x)dx \right| \leqslant \left| \int_a^A f(x, y)\,dx - \int_a^A \varphi(x)dx \right| + \left| \int_A^\infty f(x, y)dx \right| + \left| \int_A^\infty \varphi(x)dx \right|.$$

If we take a number $\varepsilon > 0$ arbitrarily, then we can first fix A so that the second and third summands on the right are $< \varepsilon/3$ (independently of y!), and then we can so approximate y to y_0 that we also make the first summand $< \varepsilon/3$ [Sec. 296]. Then, for the given y, it will be true that

$$\left| \int_a^\infty f(x, y)\,dx - \int_a^\infty \varphi(x)\,dx \right| < \varepsilon,$$

which leads to (2).

Hence, by applying Dini's generalized theorem [Sec. 295, *(3)*] we can obtain the

COROLLARY. *Let the non-negative function $f(x, y)$ be continuous with respect to x for $x \geqslant a$ and let it tend, increasing as y increases, to the limit function $\varphi(x)$, which is also continuous in the given interval. Then from the existence of the integral*

$$\int_a^\infty \varphi(x)\,dx \tag{3}$$

follows both the existence of the integral (1) *(for all y in (γ)) and of formula* (2).

By the theorem mentioned, under the given conditions the function $f(x, y)$ will tend to $\varphi(x)$ uniformly with respect to x in any finite interval. Further, by virtue of Theorem 1, Sec. 285 the integral (1) exists, because

$$f(x, y) \leqslant \varphi(x).$$

At the same time the function $\varphi(x)$ plays the part of the majorant [Sec. 302], which guarantees the uniform (with respect to y) convergence of the integral (1). Thus, all the conditions for the application of the preceding theorem are satisfied.

Usually it is necessary to discuss the suitability of taking limits under the integral sign for a sequence of functions $\{f_n(x)\}$. By passing from a sequence to an infinite series, we can thus obtain new theorems on the termwise integration of series of functions.

For instance, here is the form the corollary takes:

Let the series

$$\sum_{n=1}^{\infty} u_n(x),$$

consisting of functions which are positive and continuous for $x \geqslant a$, *have for these values of* x *a continuous sum* $\varphi(x)$. *If the latter is integrable in the interval* $[a, +\infty]$, *then in this interval the series can be integrated termwise.*

Finally, a simple corollary to Theorem 1 is a theorem on the continuity of the integral (1) with respect to the parameter.

THEOREM 2. *Let the function* $f(x, y)$ *be defined and continuous (as a function of two variables) for the values of* $x \geqslant a$ *and the values of* y *in the interval* $[c, d]$. *If the integral* (1) *converges uniformly with respect to* y *in the interval* $[c, d]$, *then it is a continuous function of the parameter* y *in this interval.*

In fact, as we saw in Sec. 296, as x varies in any finite interval $[a, A]$ the function $f(x, y)$, as $y \to y_0$ (where y_0 is any particular value of y) tends to the limit function $f(x, y_0)$ uniformly with respect to x. But then, by Theorem 1, we can take limits under the integral sign in the integral (1):

$$\lim_{y \to y_0} I(y) = \lim_{y \to y_0} \int_a^{\infty} f(x, y)\,dx = \int_a^{\infty} f(x, y_0)\,dx = I(y_0),$$

which proves our assertion.

It is easy to prove an analogue of Dini's theorem [Sec. 267]:

THEOREM 3. *If the continuous function* $f(x, y)$ *is non-negative, then from the continuity of the integral* (1) *as a function of the parameter follows its uniform convergence.*

In this case the continuous [by Theorem 2, Sec. 296] function of y

$$\int_a^A f(x, y)\,dx$$

increases as A increases and hence [by the generalization of Dini's theorem, Sec. 295, *(3)*] tends to its limit (1) uniformly with respect to y, thus proving the theorem.

305. The integration of an integral with respect to the parameter. First we prove the following theorem:

THEOREM 4. *With the hypotheses of Theorem 2 the formula*

$$\int_c^d I(y)\,dy = \int_c^d dy \int_a^\infty f(x, y)\,dx = \int_a^\infty dx \int_c^d f(x, y)\,dy \tag{4}$$

holds.

In fact, for any $A > a$

$$\int_c^d I(y)\,dy = \int_c^d dy \int_a^A f(x, y)\,dx + \int_c^d dy \int_A^\infty f(x, y)\,dx;$$

but, by Theorem 4, Sec. 298,

$$\int_c^d dy \int_a^A f\,dx = \int_a^A dx \int_c^d f\,dy,$$

so that the difference

$$\int_c^d I(y)\,dy - \int_a^A dx \int_c^d f\,dy$$

can be written in the form

$$\int_c^d I(y)\,dy - \int_c^d dy \int_a^A f\,dx = \int_c^d dy \int_A^\infty f\,dx.$$

If now—for an arbitrarily small $\varepsilon > 0$—we take A large enough that

$$\left| \int_A^\infty f\,dx \right| < \varepsilon$$

holds for all y, then the given difference will be (absolutely) less than $\varepsilon(d - c)$. In that case

$$\int_c^d I(y)\,dy = \lim_{A \to \infty} \int_a^A dx \int_c^d f\,dy,$$

which is equivalent to (4).

If we use Theorem 3 [Sec. 304], then it is easy to deduce from this the

COROLLARY. *In the case of a non-negative function the continuity of the integral* (1) *alone implies formula* (4).

Thus—with the given hypotheses—we have established the validity of two integrals, one of which extends over an infinite interval, and the other over a finite one.

In many cases it is necessary to change the order of integrals, which are both taken in infinite intervals, by means of the formula

$$\int_c^\infty dy \int_a^\infty f(x, y)\,dx = \int_a^\infty dx \int_c^\infty f(x, y)\,dy. \tag{5}$$

To justify such a change in order is often a complicated and painstaking process.

Only for a small number of cases can formula (5) be justified by general considerations; for instance:

THEOREM 5. *Let the function $f(x, y)$ be defined and continuous for $x \geqslant a$ and $y \geqslant c$ and also non-negative. We presuppose, further, that both the integrals*

$$\int_a^\infty f(x, y)\,dx, \qquad \int_c^\infty f(x, y)\,dy$$

are continuous functions, the first of y, and the second of x. Then if one of the two repeated integrals

$$\int_c^\infty dy \int_a^\infty f(x, y)\,dx, \qquad \int_a^\infty dx \int_c^\infty f(x, y)\,dy \tag{6}$$

exists, the other also exists, and the equation (5) *holds.*

We suppose that the second of the integrals (6) exists. By the preceding corollary, for any finite $C > c$ we will have

$$\int_c^C dy \int_a^\infty f\,dx = \int_a^\infty dx \int_c^C f\,dy. \tag{7}$$

The function under the integral on the right

$$\int_c^C f\,dy$$

as a function of x and C is continuous with respect to x [Theorem 2, Sec. 296] and—as C increases to infinity—it increases, and tends to the function

$$\int_c^\infty f\,dy,$$

which by hypothesis is continuous and integrable with respect to x from a to ∞. By applying to the integral on the right side of (7) the corollary of Sec. 296, as $C \to \infty$ in this integral we can take limits under the integral sign. In such a case not only the limit of the integral on the left side of (7) exists, i.e. the first of the integrals (6), but also both repeated integrals are shown to be equal.

306. The differentiation of an integral with respect to the parameter. The theorem relevant to this we will deduce from Theorem 4 of the preceding section, just as in Sec. 270 we proved Theorem 5, which was based on Theorem 4 of Sec. 269.

THEOREM 6. *Let the function $f(x, y)$, defined and continuous with respect to x for $x \geqslant a$ and y in $[c, d]$, have for the given values a derivative $f'_y(x, y)$ continuous with respect to both variables. We presuppose, further, that the integral* (1) *converges for all y in $[c, d]$, and the integral*

$$\int_a^\infty f'_y(x, y)\,dx \tag{8}$$

not only exists, but converges uniformly with respect to y in the same interval. Then for any y in $[c, d]$ the formula

$$I'(y) = \int_a^\infty f'_y(x,\ y)\,dx$$

holds.

Applying Theorem 4 to the function $f'_y(x, y)$, only replacing d by any y $(c \leqslant y \leqslant d)$, we have

$$\int_c^y dy \int_a^\infty f'_y(x, y)\,dx = \int_a^\infty dx \int_c^y f'_y(x,\ y)\,dy$$

$$= \int_a^\infty f(x,\ y)\,dx - \int_a^\infty f(x,\ c)\,dx.$$

Because the function under the integral on the left, i.e. the integral

$$\int_a^\infty f'_y(x, y)dx,$$

is a continuous function of y [by Theorem 2, Sec. 304], then the integral on the left will have precisely this as its derivative with respect to y [Sec. 183, *(12)*]. Hence, the integral (1) in the last part of the equation will have the same derivative, and so differs from the integral on the left by only a constant. The theorem has been proved.

307. A remark on integrals with finite limits. We will consider in detail the question of the properties of the integral

$$I(y) = \int_a^\infty f(x, y)dx, \tag{1}$$

for which the unique singular value of x is ∞. Detailed theory can also be developed for the cases when the singular value of x is finite, for instance, coinciding with the upper limit b of the integral

$$I(y) = \int_a^b f(x, y)dx. \tag{9}$$

In all the assertions proved ∞ can be replaced by the number b, at the same time replacing uniform convergence at $x = \infty$ by uniform convergence at $x = b$ [Sec. 303]. Here, for instance, is how Theorem 1 is rephrased:

THEOREM 1*. *Let the function $f(x, y)$, defined for $a \leqslant x < b$ and for y in $\mathcal{Y}$: (1) be continuous with respect to x and (2) as $y \to y_0$ tend to the limit function $\varphi(x)$ uniformly with respect to x in every interval $[a, b-\eta]$. If, moreover, (3) the integral* (9) *converges uniformly with respect to y in the domain $\mathcal{Y}$, then*

$$\lim_{y \to y_0} \int_a^b f(x, y)dx = \int_a^b \varphi(x)dx.$$

Of course, it may happen that two singular values of x are present, say, $x = a$ and $x = \infty$ in the same integral (1). All the assertions can also be carried over to this case: often, however, it is convenient

simply to represent the integral (1) in the form of the sum of two integrals

$$\int_a^\infty = \int_a^b + \int_b^\infty \quad (a < b < \infty)$$

and to apply the given theory to each of the summands separately.

308. The evaluation of some improper integrals. The theory of integrals depending on a parameter can usefully be applied to the evaluation of many classical improper integrals.

(1). We evaluate (Euler's) integral

$$E = \int_0^\infty \frac{x^{a-1}}{1+x}\,dx.$$

First of all, we establish for what values of the parameter a this integral in general has a meaning. The singular values are: $x = \infty$, $x = 0$ (when $a \geqslant 1$ the latter point will not be singular). We separate the integral into two parts:

$$E = E_1 + E_2 = \int_0^1 + \int_1^\infty;$$

the first exists when $a > 0$, because then—if $a < 1$—the function under the integral as $x \to 0$ will be infinitely large of order $1 - a < 1$, by comparison with $1/x$ [Sec. 290], and the second when $a < 1$: then the function under the integral as $x \to \infty$ will be infinitely small of order $2 - a > 1$, by comparison with $1/x$ [Sec. 285]. Thus, the integral under consideration converges, if $0 < a < 1$; with this hypothesis we shall turn to its evaluation.

For $0 < x < 1$ we have the expansion in a series

$$\frac{x^{a-1}}{1+x} = \sum_{\mu=0}^{\infty} (-1)^\mu x^{a+\mu-1},$$

which converges uniformly, provided only that $0 < \varepsilon \leqslant x \leqslant 1 - \varepsilon' < 1$. But the partial sum has the majorant

$$0 \leqslant \sum_{\mu=0}^{n-1} (-1)^\mu x^{a+\mu-1} = \frac{x^{a-1}[1-(-x)^n]}{1+x} < x^{a-1},$$

which is integrable in [0, 1], hence the integral of it converges uniformly both when $x = 0$ and when $x = 1$. Integrating termwise by Theorem 1 (modified), we obtain:

$$E_1 = \sum_{\mu=0}^{\infty} \int_0^1 (-1)^\mu x^{a+\mu-1}\,dx = \sum_{\mu=0}^{\infty} \frac{(-1)^\mu}{a+\mu}.$$

By the substitution $x = 1/z$ the integral E_2 takes the form

$$E_2 = \int_0^1 \frac{x^{-a}}{1+x}\,dx = \int_0^1 \frac{x^{(1-a)-1}}{1+x}\,dx.$$

Applying the expansion we obtained above, we find:

$$E_2 = \sum_{\mu=1}^{\infty} \frac{(-1)^\mu}{a-\mu}.$$

Thus,

$$E = \frac{1}{a} + \sum_{\mu=1}^{\infty} (-1)^\mu \left(\frac{1}{a+\mu} + \frac{1}{a-\mu}\right).$$

We have already met the "expansion in simple fractions" of the function $1/\sin t$ [in Sec. 293, *(3)*; see below, Sec. 406, *(3)*, Remark]:

$$\frac{1}{\sin t} = \frac{1}{t} + \sum_{\mu=1}^{\infty} (-1)^\mu \left(\frac{1}{t+\mu\pi} + \frac{1}{t-\mu\pi}\right).$$

Here letting $t = \pi a$, we arrive at the final result:

$$E = \int_0^{\infty} \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin \pi a} \qquad (0 < a < 1).$$

(2). For the evaluation of the integral

$$\int_0^{\infty} \frac{\sin x}{x}\,dx$$

[cf. Sec. 293, *(3)*] we introduce into the more general integral, containing a parameter:

$$I(a) = \int_0^{\infty} e^{-ax} \frac{\sin x}{x}\,dx \qquad (a \geqslant 0),$$

from which the preceding integral is obtained when $a = 0$. We already know that the integral $I(a)$ converges uniformly with respect to a, for $a \geqslant 0$ [Sec. 302, *(3)*], and hence—by Theorem 2, Sec. 304—is a continuous function of the parameter a for its given values. In particular,

$$I(0) = \lim_{a \to 0} I(a). \tag{10}$$

Now we will find the derivative $I'(a)$ when $a > 0$. Differentiating under the integral sign, we obtain

$$I'(a) = -\int_0^\infty e^{-ax} \sin x \, dx = -\frac{1}{1+a^2}$$

[Sec. 283, *(5)*]. This last integral converges uniformly for $a \geqslant a_0$ (where a_0 is any fixed number, greater than zero), because it is less than the convergent integral $\int_0^\infty e^{-a_0 x} dx$ [Sec. 302, *(1)*]; therefore, at least for the given values of a, we can apply Theorem 6, Sec. 306. However, as $a > 0$, we can always choose $a_0 > 0$ so that $a > a_0$. This means that the result obtained is true for any $a > 0$. In that case, for $a > 0$, we have

$$I(a) = C - \arctan a.$$

For a definite constant C we make a tend to ∞; because

$$|I(a)| \leqslant \int_0^\infty e^{-ax} dx = \frac{1}{a},$$

then here $I(a) \to 0$, and C is equal to $\pi/2$. Finally, in view of (10),

$$I(0) = \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

(3). In order to evaluate the integral

$$K = \int_0^\infty e^{-x^2} dx$$

[cf. Sec. 293, *(2)*], we let $x = ut$ it, where u is any positive number; we obtain

$$K = u \int_0^\infty e^{-u^2 t^2} dt.$$

Now we multiply both sides of this equation by $e^{-u^2}du$ and integrate with respect to u from 0 to ∞:

$$K\cdot\int_0^\infty e^{-u^2}\,du = K^2 = \int_0^\infty e^{-u^2}u\,du\int_0^\infty e^{-u^2t^2}dt.$$

It is not difficult to see that changing the order of the integrals leads very quickly to the result. In fact, after changing the order we obtain

$$K^2 = \int_0^\infty dt\int_0^\infty e^{-(1+t^2)u^2}u\,du = \frac{1}{2}\int_0^\infty \frac{dt}{1+t^2} = \frac{\pi}{4},$$

whence (because, clearly, $K>0$)

$$K = \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}.$$

In order to justify the change we have made in the order of the integrals, we will try to make use of Theorem 5, Sec. 305. But while the integral

$$\int_0^\infty e^{-(1+t^2)u^2}u\,du = \frac{1}{2}\frac{1}{1+t^2}$$

is a continuous function of t for all $t\geqslant 0$, the integral

$$\int_0^\infty e^{-(1+t^2)u^2}u\,dt = e^{-u^2}K$$

is continuous only for $u>0$, and when $u=0$ it vanishes, because it suffers a discontinuity at this point. Therefore it is impossible to apply Theorem 5 immediately to the rectangle $[0,\infty;0,\infty]$! We apply it to the rectangle $[u_0,\infty;0,\infty]$, where $u_0>0$, using the fact that the integral

$$\int_{u_0}^\infty e^{-(1+t^2)u^2}u\,du = \frac{1}{2}\frac{1}{1+t^2}e^{-(1+t^2)u_0}$$

is a continuous function of t for all $t\geqslant 0$. This justifies the equation

$$\int_{u_0}^\infty du\int_0^\infty e^{-u^2(1+t^2)}u\,dt = \int_0^\infty dt\int_{u_0}^\infty e^{-u^2(1+t^2)}u\,du.$$

It remains only, as u_0 decreases, to take limits here as $u_0 \to 0$, which, on the right side, can be done under the integral sign—on the basis of the corollary in Sec. 304.

(4). Finally, we turn to the so-called Laplace integrals†:

$$y = \int_0^\infty \frac{\cos ax}{k^2 + x^2}\, dx \quad \text{and} \quad z = \int_0^\infty \frac{x \sin ax}{k^2 + x^2}\, dx.$$

First of all, we note the uniform convergence with respect to the parameter a, and hence (by Theorem 2) the continuity with respect to a of the first of these integrals, y, which is true, moreover, for all values of $a \geqslant 0$ [see Sec. 302, *(1)*]. Considering only values of $a > 0$ and differentiating with respect to a under the integral sign (Theorem 6), we establish that

$$\frac{dy}{da} = -z; \tag{11}$$

here we are relying on the uniform convergence of the second of the integrals, z, for all $a \geqslant a_0$, where a_0 is any fixed positive number [see Sec. 302, *(2)*].

It is impossible to perform further differentiation with respect to a under the integral sign, because as a result of such differentiation a divergent integral would be obtained.

However, we add the equation (see above, (2))

$$\frac{\pi}{2} = \int_0^\infty \frac{\sin ax}{x}\, dx‡$$

termwise to the given equation and obtain:

$$\frac{dy}{da} + \frac{\pi}{2} = k^2 \int_0^\infty \frac{\sin ax}{x(k^2 + x^2)}\, dx.$$

† Pierre Simon Laplace (1749–1827) was an outstanding French astronomer, mathematician, and physicist.

‡ However, in what follows we do not need the value of this integral at all; it is sufficient simply to know that for all $a > 0$ it has a constant value; it is easy to see this by the simple substitution $t = ax$.

Here we can again differentiate under the integral sign, and by this means we find

$$\frac{d^2y}{da^2} = k^2 \int_0^\infty \frac{\cos ax}{k^2 + x^2}\, dx,$$

so that

$$\frac{d^2y}{da^2} = k^2 y.$$

For this simple linear differential equation of the second order with constant coefficients, with roots $\pm k$ of its "characteristic equation", it is easy to form the general solution

$$y = C_1 e^{ka} + C_2 e^{-ka}, \tag{12}$$

where C_1 and C_2 are constants. But for all values of a the magnitude of y is bounded:

$$|y| \leqslant \int_0^\infty \frac{dx}{k^2 + x^2} = \frac{\pi}{2k},$$

which means that necessarily $C_1 = 0$ (because otherwise as $a \to \infty$ the magnitude of y would increase infinitely). For a definite constant C_2 we set $a = 0$, because, in view of the continuity of y, the relation (12) remains in force also for this value of a; clearly:

$$C_2 = \frac{\pi}{2k}.$$

Finally,

$$y = \int_0^\infty \frac{\cos ax}{k^2 + x^2}\, dx = \frac{\pi}{2k} e^{-ka} \qquad (a \geqslant 0).$$

Hence z is also determined by differentiation [see (11)]:

$$z = \int_0^\infty \frac{x \sin ax}{k^2 + x^2}\, dx = \frac{\pi}{2} e^{-ka} \qquad (a > 0).$$

§ 4. Eulerian integrals

309. The Eulerian integral of the first type. This is what we call (after Legendre) an integral of the form

$$B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx, \tag{1}$$

where $a, b > 0$. It defines a function of two variable parameters a and b: *the B ("Beta") function.*

The integral under consideration, as we know [Sec. 290, *(3)*], converges for positive values of a and b (although they may be less than 1)† and, consequently, can really be taken as the basis for the definition of the function B. We will establish some of its properties.

(1) First of all, almost immediately (by the substitution $x = 1 - t$) we obtain:

$$B(a, b) = B(b, a),$$

so that the function B is symmetric with respect to a and b.

(2) Integrating by parts, when $b > 1$, we find from formula (1) that‡:

$$\begin{aligned} B(a, b) &= \int_0^1 (1-x)^{b-1}\,d\frac{x^a}{a} \\ &= \frac{x^a(1-x)^{b-1}}{a}\Bigg|_0^1 + \frac{b-1}{a}\int_0^1 x^a(1-x)^{b-2}\,dx \\ &= \frac{b-1}{a}\int_0^1 x^{a-1}(1-x)^{b-2}\,dx - \frac{b-1}{a}\int_0^1 x^{a-1}(1-x)^{b-1}\,dx \\ &= \frac{b-1}{a}B(a, b-1) - \frac{b-1}{a}B(a, b), \end{aligned}$$

† However, if the value of one of the parameters a, b is $\leqslant 0$, then the integral diverges.

‡ We use here the identity

$$x^a(1-x)^{b-2} = x^{a-1}(1-x)^{b-2} - x^{a-1}(1-x)^{b-1}.$$

whence

$$B(a, b) = \frac{b-1}{a+b-1} B(a, b-1). \tag{2}$$

This formula can be applied for the purpose of decreasing b, provided b remains greater than 1; thus one can always arrange that the second argument should remain $\leqslant 1$.

However, one can arrive at the same sort of relation with respect to the first argument, because—in view of the symmetry of B—the other reduction formula:

$$B(a, b) = \frac{a-1}{a+b-1} B(a-1, b) \qquad (a > 1) \tag{2'}$$

holds.

If b is equal to a natural number n, then applying formula (2) successively, we find:

$$B(a, n) = \frac{n-1}{a+n-1} \frac{n-2}{a+n-2} \cdots \frac{1}{a+1} B(a, 1).$$

But

$$B(a, 1) = \int_0^1 x^{a-1} dx = \frac{1}{a}.$$

Therefore for $B(a, n)$ and, simultaneously, for $B(n, a)$ we obtain the final expression

$$B(n, a) = B(a, n) = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)}{a \cdot (a+1) \cdot (a+2) \ldots (a+n-1)}. \tag{3}$$

If a is also equal to a natural number m, then

$$B(m, n) = \frac{(n-1)!\,(m-1)!}{(m+n-1)!}.$$

This formula can also be applied when $m = 1$ or $n = 1$, if for the symbol 0! we understand 1.

(3) For the function B we will give another analytical representation, which will often be useful. In fact, if we make the substitution $x = y/(1+y)$ in the integral (1), where y is a new variable, varying from 0 to ∞, then we obtain

$$B(a, b) = \int_0^\infty \frac{y^{a-1}}{(1+y)^{a+b-1}} dy. \tag{4}$$

Here letting $b = 1 - a$ (with the hypothesis that $0 < a < 1$), we find

$$B(a, 1-a) = \int_0^\infty \frac{y^{a-1}}{1+y}\, dy.$$

The reader already knows how to calculate the above integral, which is also associated with Euler's name [Sec. 308, *(1)*]. Substituting its value, we arrive at the formula

$$B(a, 1-a) = \frac{\pi}{\sin \pi a} \qquad (0 < a < 1). \tag{5}$$

If, in particular, we take

$$a = 1 - a = \frac{1}{2},$$

then we obtain:

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi.$$

We limit ourselves to these few properties of the "Beta" function, because—as we will see below [Sec. 311, *(4)*]—it can be expressed very simply by means of another function—the "Gamma" function, which we will consider in detail.

310. The Eulerian integral of the second type. This name was given by Legendre to the remarkable integral:

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x}\, dx, \tag{6}$$

which converges for any $a > 0$ [Sec. 290, *(4)*]† and defines the Γ ("Gamma") *function*. The Γ function, after the elementary functions, is one of the most important functions in analysis and its applications. The study of the properties of the function Γ, starting from its defining integral (6), serves at the same time as an excellent example of the application of the theory studied above of integrals depending on a parameter.

If in (6) we let

$$x = \log \frac{1}{z},$$

† When $a \leqslant 0$ the integral diverges.

then we find:

$$\Gamma(a) = \int_0^1 \left(\log \frac{1}{z}\right)^{a-1} dz.$$

As is well known [Sec. 65, *(2)*],

$$\log \frac{1}{z} = \lim_{n \to \infty} n(1 - z^{1/n}),$$

where the expression $n(1-z^{1/n})$ as n increases tends increasingly to its limit†. Then, by Secs. 304 (the Corollary) and 307

$$\Gamma(a) = \lim_{n \to \infty} n^{a-1} \int_0^1 (1 - z^{1/n})^{a-1} dz$$

or—if we perform the substitution $z = y^n$:

$$\Gamma(a) = \lim_{n \to \infty} n^a \int_0^1 y^{n-1}(1-y)^{a-1} dy.$$

But, according to (3),

$$\int_0^1 y^{n-1}(1-y)^{a-1} dy = B(n, a) = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)}{a \cdot (a+1) \cdot (a+2) \cdot \ldots \cdot (a+n-1)}.$$

Thus, finally, we arrive at the well-known Euler–Gauss formula:

$$\Gamma(a) = \lim_{n \to \infty} n^a \cdot \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1)}{a \cdot (a+1) \cdot (a+2) \cdot \ldots \cdot (a+n-1)}.$$

This formula Euler had already communicated to Goldbach in a letter in 1729, but it was forgotten. Afterwards Gauss took it as his basis for defining the function $\Pi(a) = \Gamma(a+1)$. Legendre and Lobatchevsky studied the function a great deal, and Lobatchevsky started from the original definition of the Γ function, using infinite series.

311. Some simple properties of the Γ function. *(1) The function $\Gamma(a)$, for all values of $a > 0$, is continuous and has continuous derivatives of all orders.* It is sufficient simply to prove the existence

† We can ascertain this by the methods of differential calculus, by considering the expression $(1 - z^\alpha)/\alpha$ as a function of α.

of the derivatives. Differentiating the integral (6) under the integral sign, we obtain

$$\Gamma'(a) = \int_0^\infty x^{a-1} \log x \cdot e^{-x} dx. \tag{7}$$

The application of Leibniz's rule is justified by the fact that both the integrals

$$\int_0^1 x^{a-1} \log x \cdot e^{-x} dx \quad \text{and} \quad \int_1^\infty x^{a-1} \log x \cdot e^{-x} dx$$

converge uniformly with respect to a: the first when $x = 0$ for $a \geqslant a_0 > 0$ (the majorant is $x^{a_0-1}|\log x|$), and the second when $x = \infty$ for $a \leqslant A < \infty$ (the majorant is $x^A e^{-x}$†).

In the same way we can ascertain the existence of the second derivative and all subsequent derivatives.

$$\Gamma''(a) = \int_0^\infty x^{a-1} (\log x)^2 e^{-x} dx. \tag{7*}$$

(2) From (6), integrating by parts, we obtain at once:

$$a \int_0^\infty x^{a-1} e^{-x} dx = x^a e^{-x} \Big|_0^\infty + \int_0^\infty x^a e^{-x} dx,$$

i.e.

$$\Gamma(a+1) = a\,\Gamma(a). \tag{8}$$

This formula, by repeated application, gives

$$\Gamma(a+n) = (a+n-1) \cdot (a+n-2) \cdot \ldots \cdot (a+1) \cdot a\,\Gamma(a). \tag{8*}$$

By this means the calculation of Γ for an arbitrary value of the argument a can be reduced to the calculation of Γ for $0 < a \leqslant 1$ (or, if desired, for $1 < a \leqslant 2$).

If in (8*) we take $a = 1$ and we bear in mind that

$$\Gamma(1) = \int_0^\infty e^{-x} dx = 1, \tag{9}$$

then it turns out that

$$\Gamma(n+1) = n! \tag{10}$$

† For $x > 0$, clearly, $\log x < x$.

The Γ function, thus, by a natural extension over the domain of any positive values of the argument, is the function $n!$ only for a natural value of n.

(3) The rate of variation of the Γ function. Now we can give a general representation of the behaviour of the function $\Gamma(a)$ as a increases from 0 to ∞.

From (9) and (10) we have: $\Gamma(1) = \Gamma(2) = 1$, so that, by Rolle's theorem, the root a_0 of the derivative $\Gamma'(a)$ must lie between 1 and 2. This derivative increases constantly, because the second derivative $\Gamma''(a)$, as is seen from its expression in (7*), is always positive. Hence, when $0 < a < a_0$ the derivative $\Gamma'(a) < 0$, and the function $\Gamma(a)$ decreases, and when $a_0 < a < \infty$ it will be true that $\Gamma'(a) > 0$, so that $\Gamma(a)$ increases; when $a = a_0$ there is a minimum. The calculation, which we do not quote, gives:

$$a_0 = 1.4616\ldots, \quad \min \Gamma(a) = \Gamma(a_0) = 0.8856\ldots.$$

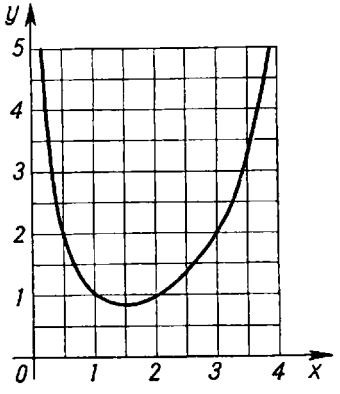

FIG. 5.

It is interesting to establish the limit for $\Gamma(a)$ as a approximates to 0 or to ∞. From (8) (and from *(1)*) it is clear that

$$\Gamma(a) = \frac{\Gamma(a+1)}{a} \to +\infty$$

as $a \to +0$. On the other hand, in view of (10),

$$\Gamma(a) > n!, \text{ provided only } a > n+1,$$

i.e. $\Gamma(a) \to +\infty$ as $a \to +\infty$.

The graph of the function $\Gamma(a)$ is shown in Fig. 5.

(4) The connection between the functions B and Γ. In order to establish this connection, we transform (6) by means of the substitution $x = ty (t > 0)$ into the form:

$$\frac{\Gamma(a)}{t^a} = \int_0^\infty y^{a-1} e^{-ty}\, dt. \tag{11}$$

Here replacing a by $a + b$ and at the same time t by $1 + t$, we obtain:

$$\frac{\Gamma(a+b)}{(1+t)^{a+b}} = \int_0^\infty y^{a+b-1} e^{-(1+t)y}\, dy.$$

Now we multiply both sides of this equation by t^{a-1} and integrate with respect to t from 0 to ∞:

$$\Gamma(a+b)\int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}}\, dt = \int_0^\infty t^{a-1}\, dt \int_0^\infty y^{a+b-1} e^{-(1+t)y}\, dy.$$

In the integral on the left we recognize the function $B(a, b)$ [see (4)]; on the right we change the order of the integrals. As a result we obtain [by using (7) and (6)]:

$$\Gamma(a+b)B(a,b) = \int_0^\infty y^{a+b-1} e^{-y}\, dy \int_0^\infty t^{a-1} e^{-ty}\, dt$$

$$= \int_0^\infty y^{a+b-1} e^{-y} \frac{\Gamma(a)}{y^a}\, dy = \Gamma(a)\int_0^\infty y^{b-1} e^{-y}\, dy = \Gamma(a)\Gamma(b),$$

whence, finally,

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{12}$$

The elegant solution given for this Eulerian relation is due to Dirichlet. However, for its foundation it relies on the validity of changing the order of the integrals.

We will do this, limiting ourselves first to the hypothesis that $a > 1, b > 1$. Then for the function

$$t^{a-1} y^{a+b-1} e^{-(1+t)y}$$

all the conditions of Theorem 5, Sec. 305 are seen to be satisfied: this function is continuous (and indeed positive) for $y \geqslant 0$ and $t \geqslant 0$, and the integrals

$$t^{a-1}\int_0^\infty y^{a+b-1}\, e^{-(1+t)y}\, dy = \Gamma(a+b)\,\frac{t^{a-1}}{(1+t)^{a+b}}$$

and

$$y^{a+b-1}\, e^{-y}\int_0^\infty t^{a-1}\, e^{-ty}\, dt = \Gamma(a)\, y^{b-1}\, e^{-y}$$

in their turn are continuous functions: the first of t when $t \geqslant 0$, the second of y for $y \geqslant 0$. Reference to the theorem justifies changing the order of the integrals, and with it justifies the formula (12) for the case $a > 1$, $b > 1$.

If it is only known that $a > 0$ and $b > 0$, then, by what has been proved, we have

$$B(a+1, b+1) = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)},$$

and hence, using the formulae (2), (2′) mentioned for the function B and (8) for the function Γ, it is easy to obtain formula (12) again, without unnecessary limits.

(5) The formula of the complement. If we set $b = 1 - a$ (assuming $0 < a < 1$) in the formula (12), then, in view of (5) and (9), we obtain the relation

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin a\pi}, \tag{13}$$

which is called the *formula of the complement.*

When $a = 1/2$ we find from this (because $\Gamma(a) > 0$):

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{14}$$

Making in the integral

$$\int_0^\infty \frac{e^{-z}}{\sqrt{z}}\, dz = \sqrt{\pi}$$

the substitution $z = x^2$, we again obtain the integral which is well known to us

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

(6) Legendre's formula. If in the integral

$$B(a,a) = \int_0^1 x^{a-1}(1-x)^{a-1}\,dx = \int_0^1 \left[\frac{1}{4} - \left(\frac{1}{2} - x\right)^2\right]^{a-1} dx$$

$$= 2\int_0^{\frac{1}{2}} \left[\frac{1}{4} - \left(\frac{1}{2} - x\right)^2\right]^{a-1} dx$$

we perform the substitution

$$\tfrac{1}{2} - x = \tfrac{1}{2}\sqrt{t},$$

then we obtain

$$B(a,a) = \frac{1}{2^{2a-1}}\int_0^1 t^{-\frac{1}{2}}(1-t)^{a-1}\,dt = \frac{1}{2^{2a-1}}\,B\left(\frac{1}{2}, a\right).$$

Replacing, in both cases, the function B by its expression (12) in terms of Γ:

$$\frac{\Gamma(a)\Gamma(a)}{\Gamma(2a)} = \frac{1}{2^{2a-1}}\,\frac{\Gamma(1/2)\Gamma(a)}{\Gamma(a+1/2)}.$$

Keeping $\Gamma(a)$ and substituting instead of $\Gamma(1/2)$ its value $\sqrt{\pi}$ [see (14)], we arrive at *Legendre's formula*:

$$\Gamma(a)\,\Gamma\left(a + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2a-1}}\,\Gamma(2a).$$

There exist many other formulae showing in more detail the properties of the function Γ. We are not able to stop to consider them here, nor the methods for the approximate calculation of the values of the function Γ and its logarithm. We confine ourselves to mentioning that Legendre, by using the properties of the function Γ and the apparatus of infinite series, formed a table of the logarithms to the base 10 of $\Gamma(a)$ for a from 1 to 2 by differences in value of 0.001, first with 7 and then with 12 decimal figures.

This new, non-elementary function Γ is just as familiar to us as the usual functions, which we call elementary.

312. Examples. Now we introduce some simple examples using the function Γ.

(1) The integral

$$\int_0^1 x^{p-1}(1-x^m)^{q-1}dx \qquad (p, q, m>0)$$

by the substitution $x^m = y$ converges at once to the elementary integral of the first type:

$$\frac{1}{m}B\left(\frac{p}{m}, q\right) = \frac{1}{m}\frac{\Gamma(p/m)\Gamma(q)}{\Gamma(p/m+q)}.$$

(2) We will evaluate the integral

$$\int_0^{\pi/2} \sin^{a-1}\varphi \cos^{b-1}\varphi\, d\varphi \qquad (a,\ b>0).$$

If we set $x = \sin\varphi$, then it becomes the integral

$$\int_0^1 x^{a-1}(1-x^2)^{b/2-1}dx.$$

Using the preceding example, we will have

$$\int_0^{\pi/2} \sin^{a-1}\varphi \cos^{b-1}\varphi\, d\varphi = \frac{1}{2}B\left(\frac{a}{2}, \frac{b}{2}\right) = \frac{1}{2}\frac{\Gamma(a/2)\Gamma(b/2)}{\Gamma[(a+b)/2]}.$$

In particular, when $b = 1$, we obtain from this

$$\int_0^{\pi/2} \sin^{a-1}\varphi\, d\varphi = \frac{\sqrt{\pi}}{2}\frac{\Gamma(a/2)}{\Gamma[(a+1)/2]}.$$

It is easy to verify that this one formula embraces both the formulae (5), Sec. 187.

If, however, in the original integral we take $a = 1+c, b = 1-c$, where $|c|<1$, then we find (by applying the formula for the complement)

$$\int_0^{\frac{\pi}{2}} \tan^c\varphi\, d\varphi = \frac{1}{2}\Gamma\left(\frac{1+c}{2}\right)\Gamma\left(\frac{1-c}{2}\right) = \frac{\pi}{2\cos c\pi/2}.$$

(3) We consider, finally, the integral

$$\int_0^{\infty} \frac{\sin^{p/q}x}{x}\, dx,$$

where p and q are relatively prime odd natural numbers.

Rewriting the integral in the form

$$\int_0^\infty \sin^{(p-q)/q} x \frac{\sin x}{x}\, dx,$$

we apply to it Lobatchevsky's general formula, mentioned on p. 134. The conditions

$$f(x+\pi) = f(x) \quad \text{and} \quad f(\pi - x) = f(x)$$

under which this formula is valid are satisfied for the function.

$$f(x) = \sin^{(p-q)/q} x.$$

Thus we obtain [see *(2)*]

$$\int_0^\infty \frac{\sin^{p/q} x}{x}\, dx = \int_0^{\pi/2} \sin^{p/q-1} x\, dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma(p/2q)}{\Gamma[(p+q)/2q]}.$$

From these few examples it should be clear to the reader to what extent—because of the function Γ we have introduced—the chances have been extended of representing integrals in a finite form by means of known functions. Sometimes, even if the function Γ does not occur in the final result, the obtaining of it is simplified by using the properties of this function.

313. Some historical remarks on changing the order of two limit operations. The purpose of this concluding section is the comparison—as an historical illustration—of all that has been said in different parts of the course in connection with *changing the order of two limit operations*. By a "limit operation" we mean here not only the immediate passage to a limit with respect to any of the arguments of the function under consideration, but also other operations which can be reduced in the final count to such a passage to a limit, namely: the summation of an infinite series, the differentiation of a function and, finally, the integration of a function between constant limits (in the proper or improper sense).

In Sec. 131 we discussed the equation in two repeated limits

$$\lim_{y \to b} \lim_{x \to a} f(x, y) = \lim_{x \to a} \lim_{y \to b} f(x, y). \tag{15}$$

We saw that such an equation does not always hold, and we established certain conditions which guarantee its validity. There was an analogous situation in Sec. 147 with respect to the equation in two mixed partial derivatives

$$D_y D_x f(x, y) = D_x D_y f(x, y). \tag{16}$$

Chapter 16 ("Sequences and Series of Functions") was mainly devoted to a problem of the type under consideration: there we considered the conditions under which it is permissible to change the order of the operation of summation of an infinite series—with the usual passage to a limit [Secs. 266, 268], differentiation [Sec. 270] and integration [Sec. 269]. Finally, we can say the same thing about the basic contents of the present chapter; only on this occasion the only operation whose order is being changed is invariably the operation of integration,

which—by observing the known conditions—we successively permute with other limit operations [Secs. 296–298, 304–307].

Long before the operations which we are discussing here were recognized as "limits" (and this process was completed, as we know, only at the beginning of the nineteenth century), the permutation of two such operations had permanently entered into mathematical practice. It is in fact realized in fundamental situations of analysis: we mention the termwise differentiation and integration of Newton, Leibniz and their contemporaries, and also "Leibniz's rule" for the differentiation of an integral with respect to a parameter. We continually meet such a change in order of taking limits throughout the eighteenth century, usually without justification, but sometimes with proofs, of course, on the level of rigorousness of the period; as an example we mention the reasoning of Euler and Claireau, justifying changing the order of two differentiations (1739). We can say that in the history of mathematical analysis changing the order of two limit operations has always served as a powerful means for obtaining many general assertions and separate mathematical facts. But when it has been applied wrongly, it has been the source of many mistakes and paradoxes. This idea, that changing the order of two limit operations is not always permissible, developed slowly—it was based on an analysis of the mistakes, and became common knowledge about the middle of the nineteenth century. A rigorous justification of the usual cases in analysis of such a change in order was only completed at about the end of the eighteenth century.

With respect to the simplest question of changing the order of two passages to limits, the situation was naturally the first to be clarified. At the end of the eighteenth or beginning of the nineteenth century it was noted for the simple examples that equation (15) might not hold, i.e. that a repeated limit sometimes depended on the order in which taking the limits was performed. In one remark of Cauchy's of 1815 (published in 1827) we find a systematic and valid presentation of this problem.

In exactly the same way as Cauchy, Gauss knew that the order of integration in a repeated integral, if the function under the integral suffered a discontinuity (for instance, if it became infinite), could not be changed without reservations. But this was far from clarifying in general the situation with respect to changing the order of limit operations.

We happened to mention [Sec. 281] Cauchy's unsuccessful attempt to prove the assertion about the continuity of the sum of a series of continuous functions and the termwise integration of such a series, which was first refuted at once by Abel, and about which Tchebychev afterwards raised objections. To what has been said here we add that in 1823 Cauchy showed to be incorrect a "proof" of the unrestricted application of "Leibniz's rule", relating to the differentiation of an integral with respect to a parameter; and this notwithstanding the fact that examples of the non-applicability of the given rule were then already known! We note that in 1828 Ostrogradski already clearly understood that—in the case when the function under the integral becomes infinite—differentiation under

the integral sign may not be permissible. Later, other authors also pointed out this fact.

For forty years, academic text books on infinitesimal analysis often gave false assertions relating to these questions. Together with these there were numerous examples emphasizing the need for care in these problems. Of these we quote only one of the earliest, mentioned by Fourier:

$$\int_0^\infty \frac{\sin at}{t}\,dt$$

[Sec. 293, Remark]. It is clear that

$$\lim_{a\to\pm 0}\int_0^\infty \frac{\sin at}{t}\,dt \neq \int_0^\infty \lim_{a\to\pm 0}\frac{\sin at}{t}\,dt = 0;$$

on the other hand, differentiation of this integral with respect to the parameter a, performed under the integral sign, leads to the result

$$\int_0^\infty \cos at\,dt,$$

which has no meaning for any value of a [Sec. 283, *(6)*].

It became necessary to make more precise the conditions under which changing the order of limit operations—of one type or another—was permissible. This clarification was assisted by the introduction at the end of forty years of the concept of uniform convergence of a series [Sec. 281] and the related concept of uniform convergence of an integral [Sec. 301]. However, this process of introducing proper rigour into the presentation of the group of problems with which we are concerned here required some ten years. Thus, for instance, we find the first rigorous justification of formula (16) given by Schwarz in 1873. Yet in 1892 the Belgian Academy awarded a prize to de la Vallée Poussin† for a work in which were given the conditions for differentiation and integration under the integral sign, extended over an infinite interval, and so on.

† Charles Jean de la Vallée Poussin (born in 1866)—a contemporary Belgian mathematician.

CHAPTER 19

IMPLICIT FUNCTIONS. FUNCTIONAL DETERMINANTS†

§ 1. Implicit functions

314. The concept of an implicit function of one variable. We presuppose that the values of two variables x and y are connected by means of an equation, which (if all its terms are put on the left) in the general case has the form

$$F(x,\ y) = 0. \tag{1}$$

Here $F(x, y)$ is a function of two variables given in some domain. If for every value of x—in some interval—there exists one or several values of y, which together with x satisfy equation (1), then this defines a one-valued or many-valued function $y = f(x)$, for which the equation

$$F\big(x,\ f(x)\big) = 0 \tag{2}$$

holds identically with respect to x.

Let us take, for instance, the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,; \tag{1a}$$

clearly it defines y as a two-valued function of x in the interval $[-a, a]$; in fact,

$$y = \pm\frac{b}{a}\sqrt{(a^2 - x^2)}.$$

And if, instead of y, we substitute this function in equation (1a), then we obtain an identity.

† A continuation of Chapters 8 and 9 of Volume 1.

Here we succeeded in finding for y a very simple analytical expression in terms of x, even in elementary functions, but such is far from being the case on most occasions.

The function $y = f(x)$ is said to be *implicit* if it is given by means of an insoluble (with respect to y) equation (1); it is *explicit* if we can examine the immediate dependence of y on x. It will be clear to the reader that these terms only characterize a way of specifying the function $y = f(x)$ and have no relation to its nature. Strictly speaking, the contrast between the implicit and explicit specifications of a function can be made quite clear only if by an explicit specification we mean an explicit analytical specification; if, however, we allow as explicit a specification with the help of any rule [Sec. 17], then the specification of the function y of x with the help of equation (1) is no worse than any other.

In the simplest case, when the equation (1) is algebraic, i.e. when the function $F(x, y)$ is a complete polynomial with respect to x and y, an indeterminant function y of x (generally many-valued) defined by it is called *algebraic*. If the degree of the equation (with respect to y) is not greater than four, then the algebraic function can be expressed explicitly in terms of radicals; if the degree is greater than four such an expression is possible only in special cases.

Now we will only be interested in the question of the existence of a single-valued "implicit" function (together with its other properties), independently of whether it can be represented in an "explicit" form by an analytical formula. However, in the stating of this problem there is nothing which is new to us; we have discussed a particular case of it when we discussed the existence and the properties of the inverse function, and the equation

$$y - f(x) = 0$$

defines the variable x as an "implicit" function of y.

The geometric treatment of the given problem is instructive. Equation (1), under known conditions, represents a plane curve [for instance, equation (1a), as is well known, represents an ellipse (Fig. 6)]; in this case it is called the *implicit equation* of the curve. The problem lies in whether the curve (1) (or part of it) can be expressed by an ordinary equation of the form $y = f(x)$, with a single-valued

function on the right; geometrically this means that the curve (or part of it) intersects a line parallel to the y axis in only one point.

If we wish to have a single-valued function, then, as we saw in the example of the ellipse, we must limit not only the domain of variation of x, but also the domain of variation of y.

For the sake of brevity, we will say that *in the rectangle* $(a, b; c, d)$ *the equation* (1) *defines* y *as a single-valued function of* x, $y = f(x)$, *if for every value of* x *in the interval* (a, b) *the equation* (1) *has one, and only one, root* $y = f(x)$ *in the interval* (c, d).

It is very important to become quite familiar with the fact that under the given conditions *the two equations*

$$F(x, y) = 0 \quad \text{and} \quad y = f(x)$$

in the rectangle $(a, b; c, d)$ *are entirely equivalent*, i.e. they are satisfied by exactly the same points in this rectangle.

Usually we will be interested in a definite point $M_0(x_0, y_0)$, satisfying equation (1) (lying on the curve), and the neighbourhood of this point will play the part of the given rectangle. Thus, for instance, in the case of the ellipse (Fig. 6), we can clearly assert

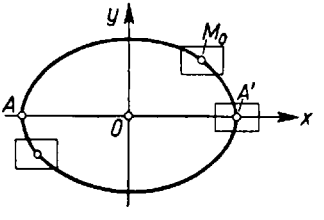

FIG. 6.

that equation (1a) defines the ordinate y as a single-valued function of the abscissa x in a sufficiently small neighbourhood of any point of the ellipse, except A, A'—the end points of its major axis.

315. The existence and properties of an implicit function. Now we will indicate some simple and easily verified conditions relating to the known function $F(x, y)$ which guarantee not only the existence of a single-valued implicit function $y = f(x)$, but also the properties of continuity and differentiability for this latter function.

THEOREM 1. *We presuppose that: (1) the function $F(x, y)$ is defined and continuous, together with its partial derivatives F'_x and F'_y, in some neighbourhood of the point $M_0(x_0, y_0)$;*

(2) $F(x, y)$ vanishes at this point: $F(x_0, y_0) = 0$, but on the contrary,

(3) the derivative $F'_y(x, y)$ is distinct from zero at this point: $F'_y(x_0, y_0) \neq 0$.

Then:

(a) *in some neighbourhood*

$$\mathcal{D}_0 = (x_0 - \delta,\ x_0 + \delta\,;\ y_0 - \delta',\ y_0 + \delta')$$

of the point $M_0(x_0, y_0)$ equation (1) *defines y as a single-valued function of x: $y = f(x)$;*

(b) *when $x = x_0$ this function takes the value y_0 : $f(x_0) = y_0$;*

(c) *in the interval $(x_0 - \delta, x_0 + \delta)$ the function $f(x)$ is continuous and*

(d) *it has a continuous derivative in this interval.*

Proof. (a) Let, say, $F'_y(x_0, y_0) > 0$. By virtue of the continuity of the derivative F'_y presupposed in *(1)*, this derivative will also be positive

$$F'_y(x,\ y) > 0$$

in a sufficiently small neighbourhood of the point (x_0, y_0)†. Let us take the closed rectangle

$$\mathcal{D} = [x_0 - \delta',\ x_0 + \delta'\,;\ y_0 - \delta',\ y_0 + \delta'],$$

lying entirely in this neighbourhood, so that the inequality given holds for all its points.

From this it follows immediately that—for any constant x from the interval $[x_0 - \delta', x_0 + \delta']$—$F(x, y)$ will be monotonic increasing as a function of y [Sec. 111].

We begin to move along the vertical passing through $M_0(x_0, y_0)$ (Fig. 7), i.e. we fix $x = x_0$; then the function under consideration $F(x, y)$ will reduce to the function $F(x_0, y)$ of one variable y. By virtue of *(2)*, when $y = y_0$ it vanishes. At the same time, as we have

† For instance, in that in which the inequality

$$|F'_y(x, y) - F'_y(x_0, y_0)| < F'_y(x_0, y_0)$$

is satisfied.

just established, the function $F(x_0, y)$ increases with y, so that for $y < y_0$ its value is less than zero, and for $y > y_0$ its value is greater than zero. Hence, in particular, it will have values of different signs at the points $A_0(x_0, y_0 - \delta')$ and $B_0(x_0, y_0 + \delta')$, in fact

$$F(A_0) = F(x_0,\ y_0 - \delta') < 0, \quad F(B_0) = F(x_0,\ y_0 + \delta') > 0.$$

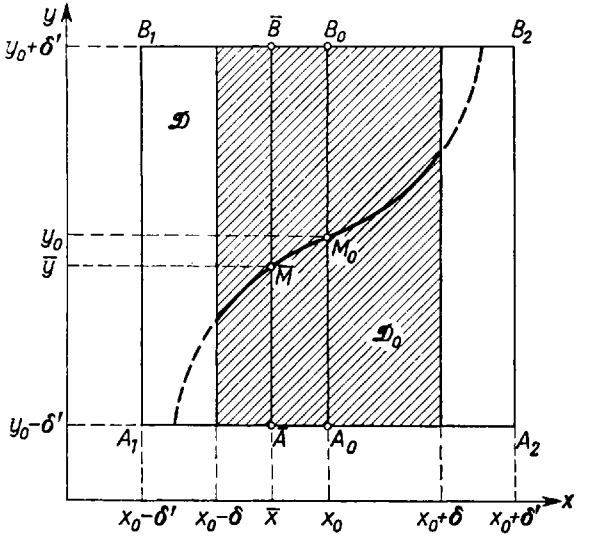

FIG. 7.

Now we turn to the horizontal lines passing through these points A_0 and B_0, i.e. this time we fix $y = y_0 - \delta'$ or $y = y_0 + \delta'$. We obtain two functions of the one variable x: $F(x, y_0 - \delta')$ and $F(x, y_0 + \delta')$, which when $x = x_0$ have a negative value and a positive value respectively. But by condition *(1)* this function is continuous†, and therefore we can find some neighbourhood $(x_0 - \delta, x_0 + \delta)$ of the point x_0 $(0 < \delta \leqslant \delta')$, in which both functions have the same sign‡, so that when $x_0 - \delta < x < x_0 + \delta$

$$F(x,\ y_0 - \delta') < 0, \quad F(x,\ y_0 + \delta') > 0.$$

† We presupposed the continuity of the function $F(x, y)$ with respect to the set of variables x, y; but in that case it will also be continuous with respect to each variable separately.

‡ This follows from Sec. 37, *(2)*. Cf. the footnote on p. 184.

In other words, on the lower and upper edges A_1A_2 and B_1B_2 of the original rectangle, along segments of length 2δ with centres at the points A_0 and B_0, the given function $F(x, y)$ has negative values on the first and positive values on the second.

In the interval $(x_0 - \delta, x_0 + \delta)$ we fix any value $x = \bar{x}$ and we consider the vertical segment joining the points $\bar{A}(\bar{x}, y_0 - \delta')$ and $\bar{B}(\bar{x}, y_0 + \delta')$. Along it our function again reduces to a function $F(\bar{x}, y)$ of one variable y. Because it is continuous† and, as we have said, at the ends of the interval $[y_0 - \delta', y_0 + \delta']$ it has values of different signs:

$$F(\bar{A}) = F(\bar{x},\ y_0 - \delta') < 0, \quad F(\bar{B}) = F(\bar{x},\ y_0 + \delta') > 0,$$

then, by the Bolzano–Cauchy theorem [Sec. 68], for some value of $y = \bar{y}$, contained between $y_0 - \delta'$ and $y_0 + \delta'$, this function $F(\bar{x}, y)$ vanishes:

$$F(\bar{x},\ \bar{y}) = 0.$$

And here, from the fact that the function $F(\bar{x}, y)$ is monotonic, it follows that when $y \gtrless \bar{y}$ we will have, respectively, $F(\bar{x}, y) \gtrless 0$, so that $\bar{y}$ is the unique value of y in the interval $(y_0 - \delta', y_0 + \delta')$, which, together with $x = \bar{x}$, satisfies equation (1). *On every vertical segment $\bar{A}\bar{B}$ we can find only one point $\bar{M}(\bar{x}, \bar{y})$ which makes the left side of the equation zero.*

Thus, in the neighbourhood

$$\mathcal{D}_0 = (x_0 - \delta,\ x_0 + \delta;\ y_0 - \delta',\ y_0 + \delta')$$

of the point $M_0(x_0, y_0)$ the equation (1), in fact, defines y as a single-valued function of x: $y = f(x)$.

(b) At the same time as the preceding reasoning, by virtue of *(2)* it is shown that $f(x_0) = y_0$. Just from the fact that $F(x_0, y_0) = 0$, we see that y_0 is the unique value of y in the interval $(y_0 - \delta', y_0 + \delta')$, which, together with $x = x_0$, satisfies equation (1).

(c) Turning to the proof of the assertions (c) and (d), we will take y to be exactly that implicit function $y = f(x)$ which is defined by equation (1) and satisfies it identically. We add to x the increment

† See page 185, footnote †.

Δx; to the increased value $x+\Delta x$ will correspond the value $y+\Delta y=f(x+\Delta x)$, which also satisfies equation (1): $F(x+\Delta x, y+\Delta y)=0$. The increment

$$\Delta F(x, y)=F(x+\Delta x, y+\Delta y)-F(x, y)=0$$

is obvious.

Now we transform this equation, making use of the formula on finite increments [Sec. 102]:

$$\begin{aligned}0&=F(x+\Delta x, y+\Delta y)-F(x, y)\\&=[F(x+\Delta x, y+\Delta y)-F(x, y+\Delta y)]\\&\quad+[F(x, y+\Delta y)-F(x, y)]\\&=F'_x(x+\theta\Delta x, y+\Delta y)\cdot\Delta x+F'_y(x, y+\theta_1\Delta y)\cdot\Delta y\\&\qquad\qquad(0<\theta, \theta_1<1),\end{aligned}$$

whence

$$\frac{\Delta y}{\Delta x}=-\frac{F'_x(x+\theta\Delta x, y+\Delta y)}{F'_y(x, y+\theta_1\Delta y)}. \tag{3}$$

First of all we will establish that as $\Delta x\to 0$ it will also be true that $\Delta y\to 0$, i.e. that the function $y=f(x)$ is continuous. We have

$$\left|\frac{\Delta y}{\Delta x}\right|=\frac{|F'_x(x+\theta\Delta x, y+\Delta y)|}{F'_y(x, y+\theta_1\Delta y)}.$$

But the function $|F'_x|$, which is continuous in $\mathcal{D}$, is bounded above by a finite number M [Sec. 137]

$$|F'_x|\leqslant M,$$

and at the same time the positive continuous function F'_y, which has a least value m in $\mathcal{D}$ [Sec. 137], also positive, is bounded below:

$$F'_y\geqslant m>0.$$

Now it is easy to obtain the approximation

$$\left|\frac{\Delta y}{\Delta x}\right|\leqslant\frac{M}{m}$$

or

$$|\Delta y|\leqslant\frac{M}{m}|\Delta x|,$$

from which follows the required assertion.

(d) Returning to equation (3), we presuppose now that $\Delta x \to 0$. Because we already know that here $\Delta y \to 0$ also, then, by the continuity of the functions F'_x and F'_y and bearing in mind that $F'_y \neq 0$, on the right of the limit we obtain

$$-\frac{F'_x(x, y)}{F'_y(x, y)}.$$

This will also be the limit of the left side of equation (3), so that the derivative of y with respect to x exists:

$$f'(x) = y'_x = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -\frac{F'_x(x, y)}{F'_y(x, y)}. \tag{4}$$

Substituting here $f(x)$ instead of y, we obtain

$$f'(x) = -\frac{F'_x(x, f(x))}{F'_y(x, f(x))};$$

in the numerator and denominator we have continuous functions of continuous functions, and the denominator does not vanish, so it is clear that $f'(x)$ is also a continuous function. The theorem has been proved completely.

Thus, from the properties of the function $F(x, y)$, which are given us immediately, we can determine the properties of the function $y = f(x)$, about which we have no immediate information.

We have already mentioned in Sec. 314 the geometric treatment of the problem studied here. If the curve is given "implicitly" by equation (1), and at any of its points $F'_y \neq 0$, then in the neighbourhood of this point the curve can be expressed "explicitly" by an equation in the form: $y = f(x)$. Of course, x and y can change roles: if at some point $F'_x \neq 0$, then in a neighbourhood of it the curve can be expressed "explicitly" by the same equation in another form: $x = g(y)$. Only at a "singular" point [Sec. 210], where simultaneously $F'_x = 0$ and $F'_y = 0$, does our theorem not apply.

316. An implicit function of several variables. Analogously to equation (1), we can consider an equation with a larger number of variables. Let us have, for instance, an equation with three variables:

$$F(x, y, z) = 0. \tag{5}$$

Under known conditions, z can be defined by this equation as an "implicit" function of the two variables x and y:

$$z = h(x, y),$$

which, generally speaking, will be many-valued. If we substitute it instead of z, then we will have

$$F(x, y, h(x, y)) = 0 \tag{6}$$

identically with respect to x, y.

We will say that *in the parallelepiped*

$$(a, b; c, d; e, f)$$

equation (5) *defines z as a single-valued function of x and y: $z = h(x, y)$, if for any point (x, y) contained in the rectangle*

$$(a, b; c, d)$$

the equation (5) *has one, and only one, root $z = h(x, y)$ in the interval (e, f).*

If this is true, then *the equations*

$$F(x, y, z) = 0 \quad \text{and} \quad z = h(x, y)$$

are completely equivalent in the parallelepiped $(a, b; c, d; e, f)$.

The part of such a parallelepiped will usually be taken by a neighbourhood of the point $P_0(x_0, y_0, z_0)$ which interests us.

Now we formulate a theorem relating to equation (5).

THEOREM 2. *We presuppose that:*

(1) the function $F(x, y, z)$ is defined and continuous, together with its partial derivatives, in some neighbourhood of the point $P_0(x_0, y_0, z_0)$;

(2) the function F vanishes at the point $P_0(x_0, y_0, z_0)$; and finally,

(3) the derivative F'_z is not equal to zero at this point.

Then:

(a) *in some neighbourhood of the point $P_0(x_0, y_0, z_0)$*

$$\mathcal{C}_0 = (x_0 - \Delta, x_0 + \Delta; y_0 - \Delta', y_0 + \Delta'; z_0 - \Delta'', z_0 + \Delta'')$$

equation (5) *defines z as a single-valued function of x and y:*

$$z = h(x, y);$$

(b) *when $x = x_0, y = y_0$ this function takes the value z_0:*

$$h(x_0, y_0) = z_0;$$

(c) *the function* $h(x, y)$ *is continuous with respect to the set of its arguments and*

(d) *it has continuous partial derivatives* h'_x, h'_y.

We will not pause to prove this, because the proof is exactly analogous to that of Theorem 1.

This time in the geometrical sense equation (5) is a surface. Under the condition $F'_z \neq 0$ in the neighbourhood of an appropriate point the surface can also be expressed by an explicit equation of the form $z = h(x, y)$. If at any point simultaneously $F'_x = F'_y = F'_z = 0$, then the theorem is not applicable—the point is "singular" [Sec. 213].

317. The determination of implicit functions from a system of equations. Finally, in the general case there may be given a system of m equations in $n + m$ variables:

$$\left.\begin{aligned} F_1(x_1, \ldots, x_n; y_1, \ldots, y_m) &= 0, \\ F_2(x_1, \ldots, x_n; y_1, \ldots, y_m) &= 0, \\ \cdots\cdots\cdots\cdots\cdots & \\ F_m(x_1, \ldots, x_n; y_1, \ldots, y_m) &= 0. \end{aligned}\right\} \tag{7}$$

Here we will discuss the determination of this system of m variables $y_1, \ldots, y_m$ as "implicit" functions of the n variables $x_1, \ldots, x_n$:

$$y_1 = f_1(x_1, \ldots, x_n), \ \ldots, \ y_m = f_m(x_1, \ldots, x_n),$$

so that by substituting them in (5) we obtain an identity with respect to $x_1, \ldots, x_n$.

We will consider the simplest case in detail—a system of two equations with three variables:

$$F(x, y, z) = 0, \quad G(x, y, z) = 0. \tag{8}$$

We say that in the parallelepiped

$$(a, b; c, d; e, f)$$

this system defines y *and* z *as single-valued functions of* x: $y = f(x)$, $z = g(x)$, *if for every value of* x *in the interval* (a, b) *the system of equations* (8) *has one, and only one, system of solutions* $(y = f(x))$, $z = h(x))$ *in the rectangle* $(c, d; e, f)$.

Under these conditions, it is clear that again *the system* (8) *within the limits for the parallelepiped is equivalent to the system*

$$y = f(x),\ z = g(x).$$

We saw that in the problem of the existence of a single-valued implicit function, defined by one equation (1) [or (5)], the decisive part was played by the requirement that at the point under consideration, satisfying the equation, the derivative F'_y [respectively, the derivative F'_z] should not vanish, in fact the derivative with respect to the variable which is subject to definition as an implicit function. However, in the problem of the existence of single-valued implicit functions y, z, defined by the system of equations (8), to which we now turn, an analogous part will be played by the determinant formed of the four partial derivatives of the functions on the left sides of (8) with respect to the variables y and z subject to definition:

$$J = J(x,\ y,\ z) = \begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}. \tag{9}$$

THEOREM 3. *We presuppose that:*

(1) the functions $F(x, y, z)$ *and* $G(x, y, z)$ *are defined and continuous, together with their partial derivatives, in some neighbourhood of the point* $P_0(x_0, y_0, z_0)$*;*

(2) the point P_0 *satisfies the system* (8)*:*

$$F(x_0, y_0, z_0) = 0, \quad G(x_0, y_0, z_0) = 0;$$

(3) the determinant $J(x, y, z)$ *is distinct from zero at this point:* $J(x_0, y_0, z_0) \neq 0$.

Then:

(a) *in some neighbourhood*

$$\mathfrak{M} = (x_0 - \delta,\ x_0 + \delta;\ y_0 - \delta',\ y_0 + \delta';\ z_0 - \delta'',\ z_0 + \delta'')$$

of the point P_0 *the system of equations* (8) *defines* y *and* z *as single-valued functions of* x*:* $y = f(x)$, $z = g(x)$*;*

(b) *when* $x = x_0$ *these functions take, respectively, the values* y_0 *and* z_0*:* $f(x_0) = y_0$, $g(x_0) = z_0$*;*

(c) *in the interval* $(x_0 - \delta,\ x_0 + \delta)$ *the functions* $f(x)$ *and* $g(x)$ *are continuous and*

(d) *they have continuous derivatives* $f'(x)$, $g'(x)$.

Proof. In so far as the determinant J is distinct from zero at the point $P_0(x_0, y_0, z_0)$, in its second column at least one element must also be distinct from zero at this point; let, for instance,

$$F'_z(x_0, y_0, z_0) \neq 0.$$

In that case, by Theorem 2, in some neighbourhood

$$\mathcal{E}_0 = (x_0 - \Delta,\ x_0 + \Delta;\ y_0 - \Delta',\ y_0 + \Delta';\ z_0 - \Delta'',\ z_0 + \Delta'')$$

of the point P_0 the first of the equations (8) defines z as a single-valued function of x and y: $z = h(x, y)$, with the properties listed in the conclusions (b), (c) and (d) of this theorem. Changing in the system (8) the first equation to the equation $z = h(x, y)$ which is equivalent to it (within the boundary of $\mathcal{E}_0$), we obtain the equivalent system

$$G(x, y, z) = 0, \quad z = h(x, y).$$

Finally, if we substitute $h(x, y)$ instead of z in G, then we arrive at the simpler, but still completely equivalent, system

$$\Phi(x, y) = 0, \quad z = h(x, y), \tag{10}$$

where for brevity we set

$$\Phi(x, y) \equiv G(x, y, h(x, y)). \tag{11}$$

Thus, we have reduced the problem to the proof of the fact that in some neighbourhood $\mathcal{M}_0$ (contained in $\mathcal{E}_0$) of the point P_0 the system of equations (10) defines y and z as single-valued functions of x, possessing all the required properties. We use the fact which is convenient for our purposes, that the first of equations (10) contains only the variables x, y, and we apply to it Theorem 1, which has already been proved; if we succeed in establishing that by this equation y is defined as a single-valued function of x: $y = f(x)$, then z can also be defined without difficulty as a single-valued function of x by means of the second of equations (10):

$$z = h(x, f(x)) = g(x). \tag{12}$$

In starting to verify that the conditions of Theorem 1 are satisfied for the function Φ, we note, first of all, that

$$h(x_0, y_0) = z_0 \tag{13}$$

[(b), Theorem 2], and by virtue of the continuity of the function $h(x, y)$ at the point $M_0(x_0, y_0)$ the value of this function near M_0 differs from z_0 by as little as desired. Then in a sufficiently small neighbourhood of the point M_0 the function $\Phi(x, y)$ will be continuous, together with its partial derivatives, because so are the functions $G(x, y, z)$ (near P_0) and $h(x, y)$ (near M_0), from which it was formed [see (11)].

In exactly the same way condition *(2)* of Theorem 1 is satisfied:

$$\Phi(x_0, y_0) = G(x_0, y_0, h(x_0, y_0)) = G(x_0, y_0, z_0) = 0,$$

by virtue of (11), (13) and hypothesis *(2)* of the present theorem.

It remains to verify that condition *(3)* of Theorem 1 is satisfied. Differentiating (11) with respect to y, we find

$$\Phi'_y(x,\ y) = G'_y + G'_z \cdot h'_y. \tag{14}$$

But h'_y can be obtained by differentiating with respect to y identity (6):

$$F'_y + F'_z \cdot h'_y = 0, \quad \text{whence} \quad h'_y = -\frac{F'_y}{F'_z}.$$

Substituting this expression in (14), we arrive at the result

$$\Phi'_y(x,\ y) = G'_y - G'_z \cdot \frac{F'_y}{F'_z} = -\frac{J}{F'_z},$$

where on the right in place of the argument z, $h(x, y)$ has been substituted everywhere. To the point $M_0(x_0, y_0)$ on the left corresponds, in view of (13), the point $P_0(x_0, y_0, z_0)$ on the right. Because, by condition *(3)*, the determinant J is distinct from zero at the point P_0, then this is true also with respect to the derivative Φ'_0 at the point M_0.

Now we are justified in applying Theorem 1 to the equation $\Phi(x, y) = 0$ and in asserting that in some neighbourhood of the point M_0:

$$(x_0 - \delta,\ x_0 + \delta;\ y_0 - \delta',\ y_0 + \delta')$$

(where $0 < \delta < \Delta$, $0 < \delta' < \Delta'$) this equation in fact defines y as a single-valued function of x: $y = f(x)$, and then (as we have already noted) formula (12) also defines z as a single-valued function of x.

As the neighbourhood $\mathcal{M}_0$, mentioned in (a), we can take the parallelepiped

$$(x_0 - \delta,\ x_0 + \delta;\ y_0 - \delta';\ y_0 + \delta',\ z_0 - \delta'',\ z_0 + \delta''),$$

by letting $\delta'' = \Delta''$. The conclusions (b), (c) and (d) of the theorem are justified by the properties of the functions $f(x)$ and $h(x, y)$, which are formulated in Theorems 1 and 2.

Here also the theorem can be given a geometrical interpretation. The system (8), generally speaking, expresses a space curve—the intersection of the two surfaces, represented by each of the equations separately. If at some point any determinant of the second order of the matrix of the partial derivatives:

$$\begin{pmatrix} F'_x, & F'_y, & F'_z \\ G'_x, & G'_y, & G'_z \end{pmatrix},$$

say, the determinant (9), is distinct from zero, then in a neighbourhood of the point under consideration two of its coordinates (in the given case: y and z) can be considered as functions of the third (x), i.e. the curve can be expressed "explicitly" by the equations $y = f(x)$, $z = g(x)$. We can apply the theorem everywhere except at a "singular" point, where all three determinants of the matrix vanish simultaneously.

An analogous theorem can be proved for the system of equations (7) in the general form by the method of mathematical induction. For this purpose the induction is performed with respect to the number of equations, just as when we reduced the case of two equations to the case of one equation. We will not pause to formulate the proof of the general theorem; we only emphasize that in the question of the existence of a system of implicit functions $y_1, \ldots, y_m$, defined by the system (7), the decisive part is played by the determinant

$$\begin{vmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} & \cdots & \frac{\partial F_2}{\partial y_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial F_m}{\partial y_1} & \frac{\partial F_m}{\partial y_2} & \cdots & \frac{\partial F_m}{\partial y_m} \end{vmatrix}$$

—it must be distinct from zero at the point whose neighbourhood we discuss.

Remark. We call the reader's attention to the "local" character of all the theorems on the existence of implicit functions: all the time we discuss only some neighbourhood of the point under consideration. But it is in such a form that these theorems are useful; for instance, for the study of the properties of the geometrical image at the given point it is quite sufficient to limit oneself at once to a neighbourhood of it.

318. The evaluation of derivatives of implicit functions. The course of the arguments, with whose help the theorems on the existence of implicit functions were established, did not always suggest a method of evaluating the derivatives (of the first order) of the implicit functions. There was no mention at all of the derivatives of higher orders. Now we will turn to a special consideration of these important problems.

We begin with the simplest case, when equation (1) is given. We will assume that the conditions of Theorem 1 are satisfied in a neighbourhood of the point under consideration; an essential part in what follows will be played by the requirement $F'_y \neq 0$.

Because the existence of the implicit function y of x and its derivative y'_x is already known in advance, the process of evaluating this derivative can be realized by means of differentiating equation (1) with respect to x

$$F(x, y) = 0,$$

which becomes an identity if we take y to be exactly the implicit function defined by this. We obtain [as we have already had in Sec. 141, *(4)*]

$$F'_x(x, y) + F'_y(x, y)y'_x = 0, \tag{15}$$

from which we return to the known formula (4)

$$y'_x = -\frac{F'_x(x, y)}{F'_y(x, y)}.$$

Now we will go further. If the function $F(x, y)$ has a continuous derivative of the second order, then the expression on the right in formula (4) can be differentiated with respect to x, hence there exists a derivative of y'_x, i.e. a second derivative y''_{x^2} of the implicit function y. Carrying out the differentiation and substituting everywhere instead of y'_x its expression (4), we find:

$$y''_{x^2} = \frac{(F''_{xy} + F''_{y^2}y'_x)F'_x - (F''_{x^2} + F''_{xy}y'_x)F'_y}{F'^2_y}$$

$$= \frac{2F'_xF'_yF''_{xy} - F'^2_yF''_{x^2} - F'^2_xF''_{y^2}}{F'^3_y};$$

whence we see that the second derivative will be a continuous function of x.

If the function $F(x, y)$ has a continuous derivative of the third order, then, clearly, there exists a third derivative of the implicit function y'''_{x^3}; its expression can again be obtained by an immediate differentiation of the expression for y''_{x^2}, etc. By mathematical induction it is easy to prove that *the existence of derivatives of the function $F(x, y)$ up to the k-th order $(k > 1)$, inclusively, guarantees the existence of the (continuous) derivative of the k-th order of the implicit function.*

When the existence of a sequence of derivatives of the implicit function has thus been established, they can be evaluated more simply by means of the repeated differentiation of the identity (15), taking into account the fact that y is a function of x. For instance, the first differentiation of this identity gives us

$$F''_{x^2} + F''_{xy}y'_x + (F''_{xy} + F''_{y^2}y'_x)y'_x + F'_y y''_{x^2} = 0,$$

whence (because $F'_y \neq 0$!)

$$y''_{x^2} = -\frac{F''_{x^2} + 2F''_{xy}y'_x + F''_{y^2}y'^2_x}{F'_y};$$

substituting for y'_x its expression (4), we return to the expression already found for y''_{x^2}, etc.

Example. (1) Let y be connected with x by the equation

$$\log \sqrt{(x^2 + y^2)} = \arctan\frac{y}{x}.$$

Differentiating successively with respect to x (where y is regarded as a function of x), we obtain

$$\frac{x + yy'}{x^2 + y^2} = \frac{xy' - y}{x^2 + y^2} \quad \text{or} \quad x + yy' = xy' - y;$$

then

$$1 + y'^2 + yy'' = xy''; \ldots.$$

From the first equation we find

$$y' = \frac{x + y}{x - y},$$

from the second (if we substitute the value we have found for y')

$$y'' = \frac{1 + y'^2}{x - y} = 2\frac{x^2 + y^2}{(x^2 - y)^2},$$

etc.

The situation is analogous in the case of equation (5)

$$F(x, y, z) = 0.$$

Here we presuppose that the conditions of Theorem 2 are satisfied. If we take z to be the implicit function defined by this equation, then it becomes an identity which can be differentiated both with respect to x, and with respect to y. As a result we obtain

$$z'_x = -\frac{F'_x}{F'_z}, \qquad z'_y = -\frac{F'_y}{F'_z}.$$

(The second equation we have really already obtained by this method in the process of proving Theorem 3.)

If the function F has continuous derivatives of the first, second, third, ... orders, then the function z also has such derivatives: all this is similar to what has already been said above about equation (1).

If all the derivatives of the first, second, third, ... order are needed, it is simpler to evaluate $dz, d^2z, \ldots$ at once. We differentiate our identity in the usual way, i.e. we equate to zero the complete differential of its left side (using here the invariance of the form of the first differential, Sec. 143):

$$\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy+\frac{\partial F}{\partial z}dz=0,$$

so that

$$dz=-\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}}dx-\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}dy.$$

At the same time

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy.$$

In view of the arbitrary nature of dx and dy it is clear† here that

$$\frac{\partial z}{\partial x}=-\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}},\qquad \frac{\partial z}{\partial y}=-\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}},$$

as we obtained above.

Differentiating once again, we obtain

$$\left[\frac{\partial^2 F}{\partial x^2}dx+\frac{\partial^2 F}{\partial x\,\partial y}dy+\frac{\partial^2 F}{\partial x\,\partial z}dz\right]dx+\ldots+\frac{\partial F}{\partial z}d^2z=0$$

and we define d^2z, which leads us to expressions for

$$\frac{\partial^2 z}{\partial x^2},\quad \frac{\partial^2 z}{\partial x\,\partial y},\quad \frac{\partial^2 z}{\partial y^2},$$

etc. We see that in all these calculations a fundamental part is played by the condition that

$$F'_z=\frac{\partial F}{\partial z}\neq 0.$$

Example. (2) Let an implicit function z of x, y be defined by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

† The equation $A\,dx+B\,dy=A'\,dx+B'\,dy$ with arbitrary values of dx and dy can only hold when $A=A'$ and $B=B'$.

We have successively

$$\frac{x\,dx}{a^2}+\frac{y\,dy}{b^2}+\frac{z\,dz}{c^2}=0, \quad dz=-\frac{c^2x}{a^2z}\,dx-\frac{c^2y}{b^2z}\,dy,$$

so that

$$\frac{\partial z}{\partial x}=-\frac{c^2x}{a^2z}, \quad \frac{\partial z}{\partial y}=-\frac{c^2y}{b^2z}.$$

Then

$$\frac{dx^2}{a^2}+\frac{dy^2}{b^2}+\frac{dz^2}{c^2}+\frac{z\,d^2z}{c^2}=0,$$

whence (if we use the known expression for dz)

$$d^2z=-\frac{c^4}{z^3}\left[\left(\frac{x^2}{a^2}+\frac{z^2}{c^2}\right)\frac{dx^2}{a^2}+\frac{2xy}{a^2b^2}\,dx\,dy+\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\frac{dy^2}{b^2}\right]$$

which gives us

$$\frac{\partial^2z}{\partial x^2}=-\frac{c^4}{a^2z^2}\left(\frac{x^2}{a^2}+\frac{z^2}{c^2}\right), \quad \frac{\partial^2z}{\partial x\,\partial y}=-\frac{c^4xy}{a^2b^2z^3},$$

$$\frac{\partial^2z}{\partial y^2}=-\frac{c^4}{b^2z^3}\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right),$$

etc.

Now we turn to the examination of the system of equations (8):

$$F(x, y, z)=0, \quad G(x, y, z)=0.$$

We will presuppose that in the neighbourhood of the point taken the conditions of Theorem 3 are satisfied. Again we call attention to the part which will be played by the requirement $J\neq 0$.

We know that the implicit functions y and z of x have derivatives with respect to x. Their evaluation is carried out by differentiating the identities which are obtained from (8), if for y and z we take precisely the given implicit functions. Differentiation with respect to x, for instance, gives

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}+\frac{\partial F}{\partial z}\frac{dz}{dx}=0,$$

$$\frac{\partial G}{\partial x}+\frac{\partial G}{\partial y}\frac{dy}{dx}+\frac{\partial G}{\partial z}\frac{dz}{dx}=0.$$

This is a system of linear equations with respect to the unknowns dy/dx and dz/dx with determinant J distinct from zero. Hence both derivatives with respect to x can easily be defined; the same is done for the derivatives with respect to y.

We will not repeat here our remarks about the use of total differentiation, the existence and evaluation of derivatives of higher powers.

Finally, all that has been said can be extended to the general case.

Example. (3) Let there be given a system

$$x+y+z+u=a, \quad x^2+y^2+z^2+u^2=b^2, \quad x^3+y^3+z^3+u^3=c^3,$$

defining y, z, u as functions of x. We have

$$1+y'+z'+u'=0, \quad x+yy'+zz'+uu'=0,$$
$$x^2+y^2y'+z^2z'+u^2u'=0.$$

Presupposing that the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ y & z & u \\ y^2 & z^2 & u^2 \end{vmatrix} = (z-y)(u-y)(u-z)$$

is not equal to zero, we have from this

$$y' = -\frac{(z-x)(u-x)}{(z-y)(u-y)}, \quad \text{etc.}$$

§ 2. Some applications of the theory of implicit functions

319. Relative extremes. We will consider the question of an extreme of the function $f(x_1, \ldots, x_{n+m})$ of $n+m$ variables with the hypothesis that these variables are subject to m "connecting equations"

$$\Phi_i(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) = 0 \quad (i = 1, 2, \ldots, m).$$

We will clarify the concept of such a relative extreme and will indicate ways of finding it.

We say that *at the point* $P_0(x_1^0, \ldots, x_{n+m}^0)$, *satisfying the connecting equations, the function* $f(x_1, \ldots, x_{n+m})$ *has a relative maximum (minimum), if the inequality*

$$f(x_1, \ldots, x_{n+m}) \underset{(\geqslant)}{\leqslant} f(x_1^0, \ldots, x_{n+m}^0)$$

is satisfied in some neighbourhood of the point M_0 *for all points* $(x_1, \ldots, x_{n+m})$, *satisfying the connecting equation.*

If, for instance, we are discussing the function $u = f(x, y, z)$ of the three variables x, y, z, subject to the connecting equation

$$F(x, y, z) = 0,$$

then finding the relative extreme of the function u means geometrically that an extreme is sought on the surface expressed by the given equation: and this extreme point and the points which are near

it must lie on this surface. If there are two connecting equations

$$F(x, y, z) = 0, \quad G(x, y, z) = 0,$$

then, clearly, the discussion simply coincides with the examination of points on the curve which is given by these equations.

Going on to a detailed study of the problem, for simplicity in writing we will limit ourselves to the function

$$u = f(x, y, z, t)$$

of four variables, regarding them as subject to the two connecting equations:

$$F(x, y, z, t) = 0, \quad G(x, y, z, t) = 0. \tag{1}$$

Let the function f have a relative extreme at the point $P_0(x_0, y_0, z_0, t_0)$. We will presuppose that both the function f and also the functions F and G have continuous partial derivatives with respect to all arguments in a neighbourhood of the point under consideration. Let, further, at least one of the second order determinants of the matrix, formed from the partial derivatives†

$$\begin{pmatrix} F'_x & F'_y & F'_z & F'_t \\ G'_x & G'_y & G'_z & G'_t \end{pmatrix},$$

be distinct from zero at the point P_0, for instance, the determinant

$$J = \begin{vmatrix} F'_z & F'_t \\ G'_z & G'_t \end{vmatrix}. \tag{2}$$

Then, if we limit ourselves to a suitable neighbourhood of the point P_0 (by a theorem analogous to Theorem 3, Sec. 317), the system (1) is equivalent to a system of the form

$$z = \varphi(x, y), \quad t = \psi(x, y), \tag{3}$$

where φ, ψ are implicit functions, defined by the system (1). In other words, the requirement that the values of the variables x, y, z, t satisfy the connecting equations (1) can be replaced by the supposition that the variables z and t represent the functions (3) of x and y. Thus, the question of a relative extreme for the function $f(x, y, z, t)$ of four variables at the point $P_0(x_0, y_0, z_0, t_0)$ coincides

† In this case we say that the matrix has rank 2 (at the point P_0).

with the question of an ordinary (absolute) extreme for the compound function of two variables

$$f(x, y, \quad \varphi(x, y), \quad \psi(x, y)) \tag{4}$$

at the point $M_0(x_0, y_0)$.

These considerations indicate a definite way of finding points which provide relative extremes of the function $f(x, y, z, t)$: if we in fact know how to solve the connecting equations, for instance, with respect to the variables z and t, and find explicit expressions for the functions (3), then the situation coincides with finding the absolute extreme for the compound function (4). Strictly speaking, we did exactly the same thing in a series of earlier solutions of problems [Secs. 153, 154], for instance, when we were looking for the greatest value of the product $xyzt$ under the condition $x+y+z+t=4c$, etc.

Now we will indicate another way of finding the point $P_0(x_0, y_0, z_0, t_0)$, not presupposing that we have explicit expressions for the (implicit) functions (3), although we will make use of the existence of these functions here.

Thus, let the function $f(x, y, z, t)$ have a relative extreme at the point P_0 or—equivalently—let the compound function (4) have an absolute extreme at the point M_0.

Thus at this point both derivatives, with respect to x and y, of the function (4) vanish, and hence, so also does its differential. By the invariance of the form of the (first) differential [Sec. 143] this condition can be written thus:

$$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz+\frac{\partial f}{\partial t}dt=0, \tag{5}$$

where by dz and dt we understand the differentials of the functions (3) at the point M_0, at the same time as the partial derivatives are evaluated at the point P_0, because

$$\varphi(x_0, y_0)=z_0, \quad \psi(x_0, y_0)=t_0. \tag{6}$$

From (5) *it is impossible, of course, to conclude that the coefficients of the differentials are equal to zero, because not all the differentials are arbitrary.* In order to reduce the situation to arbitrarily chosen differentials, i.e. to differentials dx and dy independent of the variables,

we will try to exclude from this the differentials dz and dt of the dependent variables. This is easy to do; if we differentiate the connecting equations (1) totally, taking for z and t the functions (3)†:

$$\begin{aligned} &\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy+\frac{\partial F}{\partial z}dz+\frac{\partial F}{\partial t}dt=0,\\ &\frac{\partial G}{\partial x}dx+\frac{\partial G}{\partial y}dy+\frac{\partial G}{\partial z}dz+\frac{\partial G}{\partial t}dt=0. \end{aligned} \tag{7}$$

Here, as above, in view of (6), the partial derivatives are evaluated at the point P_0. Since, by hypothesis, the determinant (2) is not zero at this point, dz and dt can be expressed linearly from this in terms of dx and dy. If we substitute these expressions in (5), then we obtain an equation of the form

$$A\,dx+B\,dy=0,$$

where A and B denote expressions, rational with respect to the partial derivatives of the functions F, G, and here taken at the point P_0. Because in this equation there occur only the differentials, dx, dy, which are independent of the variables, i.e. which are completely arbitrary numbers, then at the point M_0 we have

$$A=0, \quad B=0.$$

Together with the connecting equations this gives four equations to define the unknowns x, y, z, and t.

Of course, we have established only necessary conditions for the extreme point $P_0(x_0, y_0, z_0, t_0)$. But in such a form the conditions may still be useful for finding the greatest (or least) value of the function f under the conditions (1), if the nature of the problem is clear in advance, whether inside the domain under consideration there must exist a point where this greatest (least) value is achieved, or whether such an assumption is made in order to imply with it that the point found satisfies other conditions.

Some examples are given below, in Sec. 321.

† Strictly speaking, we differentiate the identity which is obtained from equations (1), if we substitute in them instead of z and t the implicit functions (3). We will use this convenient method of expression from now on.

320. Lagrange's method of undetermined multipliers. In the method of investigation above, symmetry with respect to the variables is destroyed: some of them are treated as independent, some as dependent, the differentials exclude some and retain others. Sometimes this produces complications in the calculation. Lagrange suggested a method by which all the variables are treated on the same footing.

We multiply equations (7), respectively, by arbitrary ("undetermined") multipliers λ, μ and add the results term by term to (5). We obtain the equation

$$\left(\frac{\partial f}{\partial x}+\lambda\frac{\partial F}{\partial x}+\mu\frac{\partial G}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}+\lambda\frac{\partial F}{\partial y}+\mu\frac{\partial G}{\partial y}\right)dy$$
$$+\left(\frac{\partial f}{\partial z}+\lambda\frac{\partial F}{\partial z}+\mu\frac{\partial G}{\partial z}\right)dz+\left(\frac{\partial f}{\partial t}+\lambda\frac{\partial F}{\partial t}+\mu\frac{\partial G}{\partial t}\right)dt=0, \quad (8)$$

where, as before, dz and dt denote the differentials of the implicit functions (3) (in the reasoning we keep the unequal treatment of the variables); the derivatives are evaluated at the point P_0.

Now we choose values of the multipliers λ and μ such that the coefficients of the dependent differentials vanish:

$$\frac{\partial f}{\partial z}+\lambda\frac{\partial F}{\partial z}+\mu\frac{\partial G}{\partial z}=0, \quad \frac{\partial f}{\partial t}+\lambda\frac{\partial F}{\partial t}+\mu\frac{\partial G}{\partial t}=0. \quad (9)$$

This can be done in so far as the determinant (2) of the system of linear equations obtained for defining λ and μ is distinct from zero. For the chosen values of the multipliers equation (8) takes the form

$$\left(\frac{\partial f}{\partial x}+\lambda\frac{\partial F}{\partial x}+\mu\frac{\partial G}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}+\lambda\frac{\partial F}{\partial y}+\mu\frac{\partial G}{\partial y}\right)dy=0. \quad (10)$$

Here again we have a situation only involving the differentials of the independent variables, therefore their coefficients must be zero, i.e. together with (9) we have

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial F}{\partial x}+\mu\frac{\partial G}{\partial x}=0, \quad \frac{\partial f}{\partial y}+\lambda\frac{\partial F}{\partial y}+\mu\frac{\partial G}{\partial y}=0. \quad (9^*)$$

Thus, to determine the four unknowns x, y, z, t, and the two multipliers λ and μ, we have just enough equations, namely the two connecting equations and the four equations (9*) and (9).

In order to simplify the writing of these equations, one usually introduces the auxiliary function

$$\Phi = f + \lambda F + \mu G;$$

then the given equations can be written in the form

$$\frac{\partial \Phi}{\partial x} = 0, \quad \frac{\partial \Phi}{\partial y} = 0; \quad \frac{\partial \Phi}{\partial z} = 0, \quad \frac{\partial \Phi}{\partial t} = 0. \tag{11}$$

They have the same appearance as the conditions for an ordinary extreme of the function F. *This must only be regarded as a hint, to simplify memorizing.*

Lagrange's method gives us the necessary conditions. In what remains here we can repeat what was said at the end of the preceding section.

Remark. In the theory we have studied, an essential part was played by the hypothesis concerning the rank of the matrix of the partial derivatives, which we used three times. In the solution of the problems by one of the methods given, in order to make certain that we did not omit any one of the points which provide the function with its relative extremes, we established as a preliminary that the given hypothesis was satisfied in fact at all those points of the domain under consideration which satisfied the connecting equations. In simplest cases we will leave this to the reader.

We turn to some examples and problems.

321. Examples and problems. *(1)* Suppose that it is required to find the extreme of the function $u = xyzt$ under the condition $x + y + z + t = 4c$; the domain of variation of the variables is defined by the inequalities $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$, $t \geqslant 0$. We have already solved this problem in Sec. 153, *(2)*, in fact eliminating t by means of the last condition.

Applying Lagrange's method to the same problem, we introduce the auxiliary function

$$\Phi = xyzt + \lambda(x + y + z + t)\dagger$$

and form the conditions:

$$\Phi'_x = yzt + \lambda = 0, \quad \ldots, \quad \Phi'_t = xyz + \lambda = 0,$$

whence

$$yzt = xzt = xyt = xyz, \quad \text{so that} \quad x = y = z = t = c.$$

† If we call to mind the role of this function, then it becomes clear that the constant summand in Φ can be omitted.

(2) We shall return to the problem of the wires most suitable to sections in an electrical network with parallel inclusions [Sec. 154, *(3)*]. Keeping the notation adopted there, we will look for the extreme of the function

$$f(q_1, q_2, \ldots, q_n) = l_1q_1 + l_2q_2 + \ldots + l_nq_n$$

under the condition that

$$F(q_1, q_2, \ldots, q_n) = \frac{\varrho l_1 J_1}{q_1} + \frac{\varrho l_2 J_2}{q_2} + \ldots + \frac{\varrho l_n J_n}{q_n} = e;$$

here we will not even begin to introduce other variables, instead of $q_1, q_2, \ldots, q_n$, as we did above, because by our new methods the problem is solved simply.

Thus, totally differentiating the equation $F = 0$, we obtain the following expression for the differential dq_n:

$$dq_n = -\frac{q_n^2}{l_n J_n}\left\{\frac{l_1 J_1}{q_1^2}dq_1 + \ldots + \frac{l_{n-1}J_{n-1}}{q_{n-1}^2}dq_{n-1}\right\}.$$

Substituting this in the equation $df = l_1dq_1 + l_2dq_2 + \ldots + l_{n-1}dq_{n-1} + l_ndq_n = 0$, we arrive at the result:

$$\left(l_1 - \frac{q_n^2}{J_n}\frac{l_1 J_1}{q_1^2}\right)dq_1 + \ldots + \left(l_{n-1} - \frac{q_n^2}{J_n}\frac{l_{n-1}J_{n-1}}{q_{n-1}^2}\right)dq_{n-1} = 0.$$

Because $dq_1, \ldots, dq_{n-1}$ are arbitrary, their coefficients are separately equal to zero, whence

$$\frac{q_1^2}{J_1} = \frac{q_2^2}{J_2} = \ldots = \frac{q_{n-1}^2}{J_{n-1}} = \frac{q_n^2}{J_n} = \lambda^2$$

and

$$q_1 = \lambda l_1 J_1, q_2 = \lambda l_2 J_2, \ldots, q_n = \lambda l_n J_n. \tag{12}$$

The factor of proportionality λ can easily be defined from the connecting equation:

$$\lambda = \frac{\varrho}{e}\sum_{i=1}^{n} l_i \sqrt{J_i}.$$

If we apply Lagrange's method, then it is necessary to construct the auxiliary function†

$$\Phi(q_1, q_2, \ldots, q_n) = l_1q_1 + \ldots + l_nq_n + \lambda^2\left(\frac{l_1 J_1}{q_1} + \ldots + \frac{l_n J_n}{q_n}\right)$$

† For convenience we take the "undetermined multiplier" in the form λ^2 and include in it the constant ϱ.

and to set its differentials equal to zero:

$$\frac{\partial \Phi}{\partial q_1} = l_1 - \frac{\lambda^2 l_1 J_1}{q_1^2} = 0, \ \ldots, \ \frac{\partial \Phi}{\partial q_n} = l_n - \frac{\lambda^2 l_n J_n}{q_n^2} = 0,$$

whence again we obtain (12), etc.

(3) As a more complicated example we will consider the following problem: the three-dimensional ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ $(a > b > c)$ is intersected by the plane $lx + my + nz = 0$, passing through its centre; it is required to define the semi-axes of the ellipse obtained by taking this section. In other words, it is necessary to find the extreme values of the function $r^2 = x^2 + y^2 + z^2$, if the variables are subject to the two connecting equations given above.

The method of eliminating the dependent variables in Sec. 319 leads here to complicated calculations; therefore we use Lagrange's method.

In order to ascertain that the rank of the matrix

$$\begin{pmatrix} \frac{x}{a^2} & \frac{y}{b^2} & \frac{z}{c^2} \\ l & m & n \end{pmatrix}$$

is equal to 2 at all points of the intersection of the ellipsoid with the plane†, we suppose the opposite. From the vanishing of all the determinants of the second order follows the fact that the elements of the upper and lower rows are proportional; but then the equation $lx + my + nz = 0$ implies $(x/a)^2 + (y/b)^2 + (z/c)^2 = 0$, which is impossible.

Constructing the auxiliary function

$$\Phi(x, y, z) = x^2 + y^2 + z^2 + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) + 2\mu(lx + my + nz),$$

we set its derivatives equal to zero:

$$x + \lambda\frac{x}{a^2} + \mu l = 0, \quad y + \lambda\frac{y}{b^2} + \mu m = 0, \quad z + \lambda\frac{z}{c^2} + \mu n = 0. \qquad (13)$$

Multiplying these equations, respectively, by x, y, z, and adding them together we obtain (taking into account the connecting equations) that $\lambda = -r^2$.

If we presuppose, in order to be specific, that not one of the numbers l, m, n is equal to zero, then from (13) we can deduce that r is equal to neither a, b, nor c. Then equations (13) can be rewritten in the form:

$$x = -\mu\frac{la^2}{a^2 - r^2}, \quad y = -\mu\frac{mb^2}{b^2 - r^2}, \quad z = -\mu\frac{nc^2}{c^2 - r^2}.$$

† See the remark in Sec. 320.

Hence it is easy to find μ, and with it x, y, z; also, by adding together these equations, previously multiplied by l, m, n, we obtain the equation:

$$\frac{l^2a^2}{a^2-r^2}+\frac{m^2b^2}{b^2-r^2}+\frac{n^2c^2}{c^2-r^2}=0,$$

from which the two extreme values of r^2 which concern us can immediately be defined.

Because the existence of these extreme values is known in advance, we have, in this way, obtained a complete solution of the problem.

322. The concept of the independence of functions. We will consider the system of functions

$$\left.\begin{aligned} y_1 &= f_1(x_1, x_2, \ldots, x_n),\\ y_2 &= f_2(x_1, x_2, \ldots, x_n),\\ &\cdots\cdots\cdots\\ y_m &= f_m(x_1, x_2, \ldots, x_n) \end{aligned}\right\} \tag{14}$$

defined and continuous, together with their partial derivatives, in some n-dimensional open domain $\mathcal{D}$.

We can assume that one of them, for instance y_j, is a function of the remaining ones:

$$y_j = \Phi(y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_m)^{\dagger}, \tag{15}$$

where Φ is also presupposed to be a continuous function of all the arguments with continuous partial derivatives, in an $(m-1)$-dimensional domain $\mathcal{E}$, containing all possible systems of values $y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_m$, which these functions take, where the n-dimensional point $(x_1, x_2, \ldots, x_n)$ varies over the domain $\mathcal{D}$. Here we assume that the equation (15) is satisfied identically with respect to $x_1, x_2, \ldots, x_n$ in the domain $\mathcal{D}$. Then we say that in this domain the function y_j depends on the remaining ones. In particular, this will happen if y_j coincides with a constant: in this case we can set $\Phi = \text{const}$.

The functions $y_1, y_2, \ldots, y_m$ are generally called dependent in the domain $\mathcal{D}$, if one of them (and so all of them) depends on the remaining ones.

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† It is essential that the function Φ should not contain, amongst its immediate arguments, the xth.

If an identity of the form (15) does not hold in $\mathcal{D}$, nor in any partial domain contained in it, then the functions $y_1, y_2, \ldots, y_m$ are said to be independent in the domain $\mathcal{D}$.

The answer to the question of the independence of the functions is given by the consideration of the so-called functional matrix, formed from the partial derivatives of these functions with respect to all the independent variables:

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix}. \tag{16}$$

Presupposing $n \geqslant m$, first of all we have the following theorem:

THEOREM 1. *If at least one determinant of the m-th order formed from the elements of the matrix* (16) *is distinct from zero in the domain* $\mathcal{D}$, *then in this domain the functions* $y_1, y_2, \ldots, y_m$ *are independent.*

Proof. Let

$$\begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_m} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_m} \end{vmatrix} \neq 0. \tag{17}$$

If it were not this determinant which was not equal to zero, but one of the others, then, by changing the numbering of the variables, we could reduce the problem to the case (17).

We will prove the theorem by *reductio ad absurdum*. We presuppose that one of the functions, for instance y_m, can be expressed in terms of the rest, so that

$$y_m = \Phi(y_1, y_2, \ldots, y_{m-1}), \tag{18}$$

in some part $\mathcal{D}_0$ of the domain $\mathcal{D}$.

Differentiating this identity with respect to each of the variables $x_i (i = 1, 2, \ldots, m)$, we obtain a series of identities (in $\mathcal{D}_0$) of the form

$$\frac{\partial y_m}{\partial x_i} = \frac{\partial y_m}{\partial y_1}\frac{\partial y_1}{\partial x_i} + \frac{\partial y_m}{\partial y_2}\frac{\partial y_2}{\partial x_i} + \ldots + \frac{\partial y_m}{\partial y_{m-1}}\frac{\partial y_{m-1}}{\partial x_i} \quad (i = 1, 2, \ldots, m).$$

We see that the elements of the last row of the determinant (17) can be obtained by adding together the corresponding elements of the first $m - 1$ rows, previously multiplied by the factors $\partial y_m/\partial y_1, \ldots, \partial y_m/\partial y_{m-1}$. Such a determinant, as we know, is equal to zero. This contradicts the hypothesis of the theorem. The contradiction we have obtained proves the impossibility of equation (18).

323. The rank of a functional matrix. Proceeding to the general case, we introduce the following definition. We will call the rank of the functional matrix (16) (in the domain $\mathcal{D}$) the greatest of the orders of the determinants formed from the rows and columns of this matrix, and not vanishing identically in $\mathcal{D}$. It may, of course, happen that all the elements of the matrix (16) vanish identically (then we say that the rank of the matrix is 0), but this case is not interesting, because here all the functions

$$y_1, y_2, \ldots, y_m$$

simply reduce to constants. If the rank of the matrix (16) is $\mu \geqslant 1$, then there exists at least one determinant of the μth order formed from the elements of the matrix (this, of course, presupposes $m \geqslant \mu$ and $n \geqslant \mu$) and not identically equal to zero in $\mathcal{D}$, while all determinants of order greater than μ (if any such exist) are identically equal to zero. We say that the rank μ of the matrix is achieved at the point P_0, if the given determinant of the μth order is distinct from zero at this point.

THEOREM 2. *Let the rank of the functional matrix* (16) *in the domain* $\mathcal{D}$ *be* $\mu \geqslant 1$, *and let it be achieved at the point* M_0. *Then in some neighbourhood* $\mathcal{D}_0$ *of this point* μ *functions, from amongst our* m *functions, will be independent, and the remaining* $m - \mu$ *will depend on them.* [Here the independent functions will be precisely those whose derivatives occur in the determinant which is distinct from zero at the point M_0.]

Proof. For simplicity in writing, we will carry out the proof for two special cases. We presuppose that there are given three functions of three variables:

$$y_1 = f_1(x_1, x_2, x_3), \quad y_2 = f_2(x_1, x_2, x_3), \quad y_3 = f_3(x_1, x_2, x_3), \tag{19}$$

where the rank of the functional matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{pmatrix} \tag{16*}$$

is equal to two and is achieved at the point $M_0(x_1^0, x_2^0, x_3^0)$. At this point let, say, the determinant

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{vmatrix} \tag{20}$$

be distinct from zero. In view of the continuity of the partial derivatives, this will also be true in some neighbourhood of the point M_0, so that, by Theorem 1, the functions y_1 and y_2 will be independent in this neighbourhood.

We set

$$f_1(x_1^0, x_2^0, x_3^0) = y_1^0, \quad f_2(x_1^0, x_2^0, x_3^0) = y_2^0$$

and apply to the system of equations

$$f_1(x_1, x_2, x_3) - y_1 = 0, \quad f_2(x_1, x_2, x_3) - y_2 = 0 \tag{21}$$

in the five variables x_1, x_2, x_3, y_1, y_2, in the neighbourhood of the point $P_0(x_1^0, x_2^0, x_3^0, y_1^0, y_2^0)$, which satisfies this system, a theorem analogous to Theorem 3, Sec. 317. That is to say, using the fact that the determinant formed from the partial derivatives of the left-hand sides of equations (21) with respect to x_1 and x_2 is distinct from zero at the point P_0, we can assert that in some neighbourhood

$$\mathcal{U}_0 = (x_1^0 - \delta_1, x_1^0 + \delta_1; x_2^0 - \delta_2, x_2^0 + \delta_2; x_3^0 - \delta_3, x_3^0 + \delta_3; y_1^0 - \Delta_1, y_1^0 + \Delta_1; y_2^0 - \Delta_2, y_2^0 + \Delta_2)$$

of this point the system (21) defines x_1 and x_2 precisely as single-valued functions of x_3, y_1 and y_2:

$$x_1 = \varphi_1(x_3,\ y_1,\ y_2), \quad x_2 = \varphi_2(x_3,\ y_1,\ y_2). \tag{22}$$

We recall that—if we confine ourselves to the domain $\mathcal{M}_0$—the systems (21) and (22) are exactly equivalent: the points in the domain under consideration satisfying one of these systems satisfy the other. From the same theorem, on which we have based this, it follows that if instead of x_1 and x_2 we substitute the functions (22) in (21), then we obtain an identity with respect to the variables x_3, y_1, y_2 in the parallelepiped

$$(x_3^0 - \delta_3,\ x_3^0 + \delta_3;\ y_1^0 - \Delta_1,\ y_1^0 + \Delta_1;\ y_2^0 - \Delta_2,\ y_2^0 + \Delta_2).$$

The following important point now arises: *if instead of* y_1 *and* y_2 *we substitute in* (22) *the functions* f_1 *and* f_2, *then we obtain an identity with respect to the variables* x_1, x_2, x_3—at least in some neighbourhood of the point $M_0(x_1^0, x_2^0, x_3^0)$. In fact, it is sufficient to choose this neighbourhood

$$\mathcal{D}_0 = (x_1^0 - \bar{\delta}_1,\ x_1^0 + \bar{\delta}_1;\ x_2^0 - \bar{\delta}_2,\ x_2^0 + \bar{\delta}_2;\ x_3^0 - \bar{\delta}_3,\ x_3^0 + \bar{\delta}_3)$$

in such a way that:

$$0 < \bar{\delta}_1 \leqslant \delta_1, \quad 0 < \bar{\delta}_2 \leqslant \delta_2, \quad 0 < \bar{\delta}_3 \leqslant \delta_3,$$

and, moreover, so that for its points the values of y_1 and y_2, defined from (21), i.e. the values of the functions f_1 and f_2, differ from y_1^0 and y_2^0, respectively, by less than Δ_1 and Δ_2†. In fact, when the point $(x_1, x_2, x_3, y_1, y_2)$ lies within $\mathcal{M}_0$, together with equations (21) they must also satisfy equations (22).

Now we turn to the third function

$$y_3 = f_3(x_1,\ x_2,\ x_3);$$

substituting here, instead of x_1 and x_2, the functions (22), we obtain

$$y_3 = f_3(\varphi_1(x_3,\ y_1,\ y_2),\ \varphi_2(x_3,\ y_1,\ y_2),\ x_3) \equiv \Phi(x_3,\ y_1,\ y_2). \tag{23}$$

On the basis of the remark made above, if in this equation, instead of y_1, y_2 and y_3, we substitute, respectively, the functions f_1, f_2 and f_3, then it is satisfied identically for all x in the domain $\mathcal{D}_0$.

† This can be realized in view of the continuity of the functions f_1 and f_2, which take the values y_1^0 and y_2^0 at the point M_0.

In order to ascertain the dependence of the function y_3 on the functions y_1 and y_2, it only remains to prove that the function Φ in (23) does not in fact contain x_3, so that (23) can be written in the form:

$$y_3 = \Phi(y_1, y_2).$$

For this purpose, clearly, it is sufficient to establish that

$$\frac{\partial \Phi}{\partial x_3} = 0 \tag{24}$$

identically with respect to x_3, y_1, y_2. With the same definition of Φ:

$$\frac{\partial \Phi}{\partial x_3} = \frac{\partial f_3}{\partial x_1}\frac{\partial \varphi_1}{\partial x_3} + \frac{\partial f_3}{\partial x_2}\frac{\partial \varphi_2}{\partial x_3} + \frac{\partial f_3}{\partial x_3}. \tag{25}$$

On the other hand, if we differentiate with respect to x_3 the equations (21), regarding x_1 and x_2 as functions of x_3, y_1, y_2, then we obtain the equations

$$\left.\begin{aligned} \frac{\partial f_1}{\partial x_1}\frac{\partial \varphi_1}{\partial x_3} + \frac{\partial f_1}{\partial x_2}\frac{\partial \varphi_2}{\partial x_3} + \frac{\partial f_1}{\partial x_3} = 0, \\ \frac{\partial f_2}{\partial x_1}\frac{\partial \varphi_1}{\partial x_3} + \frac{\partial f_2}{\partial x_2}\frac{\partial \varphi_2}{\partial x_3} + \frac{\partial f_2}{\partial x_3} = 0, \end{aligned}\right\} \tag{26}$$

which are linear with respect to the quantities $\partial\varphi_1/\partial x_3$ and $\partial\varphi_2/\partial x_3$. The third linear equation follows, as a corollary, from the linear equations (26):

$$\frac{\partial f_3}{\partial x_1}\frac{\partial \varphi_1}{\partial x_3} + \frac{\partial f_3}{\partial x_2}\frac{\partial \varphi_2}{\partial x_3} + \frac{\partial f_3}{\partial x_3} = 0, \tag{26*}$$

therefore the determinant of the third order formed from the coefficients of the given quantities and from the free term

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{vmatrix}$$

is identically equal to zero (because the rank of the matrix (16) is two). Comparison of the equation (26*) with (25) leads to the required identity (24).

Example. We consider the system of functions

$$y_1 = x_1 + x_2 + x_3, \quad y_2 = x_1^2 + x_2^2 + x_3^2, \quad y_3 = x_1x_2 + x_2x_3 + x_3x_1.$$

It is easy to verify that the functional matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 2x_1 & 2x_2 & 2x_3 \\ x_2 + x_3 & x_3 + x_1 & x_1 + x_2 \end{pmatrix}$$

will have rank 2 in the whole space. Thus, one of the functions will depend on the other two, for instance

$$y_2 = y_1^2 - 2y_3.$$

§ 3. Functional determinants and their formal properties

324. Functional determinants. In the preceding paragraphs important tools in our investigations were determinants of a special kind, viz. those consisting of partial derivatives; they played the same sort of role in their relation to systems of functions as that played by an ordinary derivative for one function [cf., for instance, the text of Theorems 1 and 3, Secs. 315, 317]. Here we will study some of their formal properties (also recalling the corresponding properties of ordinary derivatives), which will simplify the use of the given determinants in what follows.

Let there be given n functions of n variables

$$\left.\begin{aligned} y_1 &= f_1(x_1, x_2, \ldots, x_n), \\ y_2 &= f_2(x_1, x_2, \ldots, x_n), \\ &\cdots\cdots\cdots \\ y_n &= f_n(x_1, x_2, \ldots, x_n), \end{aligned}\right\} \tag{1}$$

which are defined in some n-dimensional domain $\mathcal{D}$ and have continuous partial derivatives in it with respect to all the variables. We form the determinant of these derivatives

$$\begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_1}{\partial x_n} & \frac{\partial y_2}{\partial x_n} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}.$$

This determinant is called the functional determinant of the system of functions (1) with respect to the variables $x_1, x_2, \ldots, x_n$ and for simplicity it is denoted by the symbol

$$\frac{D(y_1, y_2, \ldots, y_n)}{D(x_1, x_2, \ldots, x_n)},$$

which resembles the notation for an ordinary derivative.

Functional determinants were made known at approximately the same time (at the end of 1833 and the beginning of 1834) by Jacobi and Ostrogradski, in connection with the same problem—the changing of variables in multiple integrals [see below, Secs. 355, 359 and 384]. Somewhat later (in 1841) Jacobi published a work in which he studied the properties of these determinants and gave in a detailed form their applications both to the problem mentioned and to the theory of implicit functions. Because of this functional determinants are usually called *Jacobi's determinants* or *Jacobians*. The notation for functional determinants, used above, was afterwards suggested by the English mathematician Duncan.

325. The multiplication of functional determinants. For simplicity in writing we will limit ourselves, for example, to determinants of the third order: the results obtained will have a completely general nature. Thus, we start from the system of functions

$$\left.\begin{aligned} y_1 &= f_1(x_1, x_2, x_3), \\ y_2 &= f_2(x_1, x_2, x_3), \\ y_3 &= f_3(x_1, x_2, x_3), \end{aligned}\right\} \qquad (1^*)$$

which are defined in the (three-dimensional) domain $\mathcal{D}$ and have continuous partial derivatives in it with respect to all the variables. In addition to the system of functions (1*), we will take the system of functions

$$\left.\begin{aligned} x_1 &= \varphi_1(t_1, t_2, t_3), \\ x_2 &= \varphi_2(t_1, t_2, t_3), \\ x_3 &= \varphi_3(t_1, t_2, t_3), \end{aligned}\right\} \qquad (2)$$

defined and having continuous derivatives in the (three-dimensional) domain $\mathcal{P}$. As the point (t_1, t_2, t_3) varies in $\mathcal{P}$ let the corresponding point (x_1, x_2, x_3) not lie outside the domain $\mathcal{D}$, so that y_1, y_2, y_3 can be considered as compound functions of t_1, t_2, t_3 by means of x_1, x_2, x_3. Now we multiply the functional determinant of the system (1*)

$$\begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

by the functional determinant of the system (2)

$$\begin{vmatrix} \frac{\partial x_1}{\partial t_1} & \frac{\partial x_1}{\partial t_2} & \frac{\partial x_1}{\partial t_3} \\ \frac{\partial x_2}{\partial t_1} & \frac{\partial x_2}{\partial t_2} & \frac{\partial x_2}{\partial t_3} \\ \frac{\partial x_3}{\partial t_1} & \frac{\partial x_3}{\partial t_2} & \frac{\partial x_3}{\partial t_3} \end{vmatrix}.$$

Here we use the well-known theorem on the multiplication of determinants, which is expressed by the formula

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix},$$

where

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + a_{i3}b_{3k} \qquad (i, k = 1, 2, 3)$$

(it is multiplied "row by column"). In our case

$$c_{ik} = \frac{\partial y_i}{\partial x_1}\frac{\partial x_1}{\partial t_k} + \frac{\partial y_i}{\partial x_2}\frac{\partial x_2}{\partial t_k} + \frac{\partial y_i}{\partial x_3}\frac{\partial x_3}{\partial t_k},$$

i.e.—by the formula for the derivative of a compound function—

$$c_{ik} = \frac{\partial y_i}{\partial t_k} \qquad (i, k = 1, 2, 3).$$

Finally, as a result we obtain the determinant

$$\begin{vmatrix} \frac{\partial y_1}{\partial t_1} & \frac{\partial y_1}{\partial t_2} & \frac{\partial y_1}{\partial t_3} \\ \frac{\partial y_2}{\partial t_1} & \frac{\partial y_2}{\partial t_2} & \frac{\partial y_2}{\partial t_3} \\ \frac{\partial y_3}{\partial t_1} & \frac{\partial y_3}{\partial t_2} & \frac{\partial y_3}{\partial t_3} \end{vmatrix},$$

but this is the functional determinant of the system of functions y_1, y_2, y_3 with respect to the variables t_1, t_2, t_3. Thus, in the abbreviated notation:

$$\frac{D(y_1, y_2, y_3)}{D(t_1, t_2, t_3)} = \frac{D(y_1, y_2, y_3)}{D(x_1, x_2, x_3)} \frac{D(x_1, x_2, x_3)}{D(t_1, t_2, t_3)}. \tag{3}$$

If we had one function y of x, where x is in turn a function of one variable t, then we would obtain the usual formula for the derivative of a compound function: $dy/dt = (dy/dx)(dx/dt)$; the formula (3) we have obtained is a generalization of it.

We note the special case when the variables t_1, t_2, t_3 are identical with y_1, y_2, y_3, so that the system of functions (2) is simply a result of "inverting" the system (1) (we suppose here that such an inversion is possible). Then the relation found above reduces to the following:

$$\frac{D(y_1, y_2, y_3)}{D(x_1, x_2, x_3)} \frac{D(x_1, x_2, x_3)}{D(y_1, y_2, y_3)} = 1 \tag{4}$$

or

$$\frac{D(y_1, y_2, y_3)}{D(x_1, x_2, x_3)} = \frac{1}{\dfrac{D(x_1, x_2, x_3)}{D(y_1, y_2, y_3)}}. \tag{5}$$

In this form it is the formula for the derivative of the inverse function.

326. The multiplication of non-square functional matrices. Here again we will clarify the general result by an example of the multiplication of rectangular matrices of a special form. We consider first the numerical matrices

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}.$$

As is well known, the square matrix

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

is called their product, where

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + a_{i3}b_{3k} \qquad (i, k = 1, 2).$$

The determinant of the second order, corresponding to this matrix:

$$\begin{vmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} \end{vmatrix},$$

is equal to the sum of the products taken by pairs of the determinants of the second order, corresponding to the matrices which are being multiplied together:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} b_{21} & b_{22} \\ b_{31} & b_{32} \end{vmatrix} + \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix} \cdot \begin{vmatrix} b_{31} & b_{32} \\ b_{11} & b_{12} \end{vmatrix}.$$

By expanding the determinants we can easily verify this identity at once (in higher algebra, the general theorem relating to the multiplication of rectangular matrices is established).

Now we apply this result to functional matrices. Let us consider two functions y_1, y_2 of three variables x_1, x_2, x_3:

$$y_1 = f_1(x_1, x_2, x_3), \quad y_2 = f_2(x_1, x_2, x_3),$$

where, in turn, the variables x_1, x_2, x_3 are functions of two variables t_1, t_2:

$$x_1 = \varphi_1(t_1, t_2), \quad x_2 = \varphi_2(t_1, t_2), \quad x_3 = \varphi_3(t_1 t_2).$$

Presupposing the existence of continuous partial derivatives for all the functions, we will try to find an expression for the functional determinant of the system of functions y_1 and y_2 with respect to the variables t_1 and t_2. For this purpose we multiply together the two functional matrices

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{\partial x_1}{\partial t_1} & \frac{\partial x_1}{\partial t_2} \\ \frac{\partial x_2}{\partial t_1} & \frac{\partial x_2}{\partial t_2} \\ \frac{\partial x_3}{\partial t_1} & \frac{\partial x_3}{\partial t_2} \end{pmatrix}.$$

In the given case

$$a_{ij} = \frac{\partial y_i}{\partial x_j} \qquad (i = 1, 2; \quad j = 1, 2, 3)$$

$$b_{jk} = \frac{\partial x_j}{\partial t_k} \qquad (j = 1, 2, 3; \ k = 1, 2)$$

and, finally,

$$c_{ik} = \frac{\partial y_i}{\partial x_1}\frac{\partial x_1}{\partial t_k} + \frac{\partial y_i}{\partial x_2}\frac{\partial x_2}{\partial t_k} + \frac{\partial y_i}{\partial x_3}\frac{\partial x_3}{\partial t_k} = \frac{\partial y_i}{\partial t_k}.$$

Applying here the result given above, we obtain the identity

$$\begin{vmatrix} \frac{\partial y_1}{\partial t_1} & \frac{\partial y_1}{\partial t_2} \\ \frac{\partial y_2}{\partial t_1} & \frac{\partial y_2}{\partial t_2} \end{vmatrix} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x_1}{\partial t_1} & \frac{\partial x_1}{\partial t_2} \\ \frac{\partial x_2}{\partial t_1} & \frac{\partial x_2}{\partial t_2} \end{vmatrix} + \begin{vmatrix} \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x_2}{\partial t_1} & \frac{\partial x_2}{\partial t_2} \\ \frac{\partial x_3}{\partial t_1} & \frac{\partial x_3}{\partial t_2} \end{vmatrix} + \begin{vmatrix} \frac{\partial y_1}{\partial x_3} & \frac{\partial y_1}{\partial x_1} \\ \frac{\partial y_2}{\partial x_3} & \frac{\partial y_2}{\partial x_1} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x_3}{\partial t_1} & \frac{\partial x_3}{\partial t_2} \\ \frac{\partial x_1}{\partial t_1} & \frac{\partial x_1}{\partial t_2} \end{vmatrix},$$

which in the abbreviated notation can be written in the form:

$$\frac{D(y_1, y_2)}{D(t_1, t_2)} = \frac{D(y_1, y_2)}{D(x_1, x_2)}\frac{D(x_1, x_2)}{D(t_1, t_2)} + \frac{D(y_1, y_2)}{D(x_2, x_3)}\frac{D(x_2, x_3)}{D(t_1, t_2)} + \frac{D(y_1, y_2)}{D(x_3, x_1)}\frac{D(x_3, x_1)}{D(t_1, t_2)}. \quad (6)$$

If we were discussing one function y, which depends on one variable t by means of three variables x_1, x_2, x_3, then we would obtain the usual formula for the differentiation of a compound function:

$$\frac{dy}{dt} = \frac{\partial y}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial y}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial y}{\partial x_3}\frac{dx_3}{dt}.$$

Formula (6) is analogous to it.

In what follows we will study in detail the analogy between derivatives and functional determinants [Sec. 354, *(2)*; Sec. 383].

CHAPTER 20

CURVILINEAR INTEGRALS

§ 1. Curvilinear integrals of the first kind

327. The definition of a curvilinear integral of the first kind. In order to arrive at this new concept in a natural way, we will consider one mechanical example which leads to it.

Let there be given a plane simple† rectifiable curve (K) (Fig. 8), along which are situated masses, where their linear density $\varrho(M)$ is known at all points of the curve. It is required to determine the mass m of the whole curve (K).

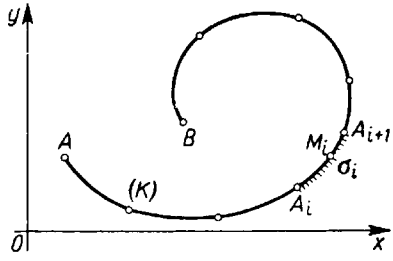

FIG. 8.

For this purpose, between the end-points A and B of the curve we place an arbitrary series of points $A_1, A_2, \ldots, A_{n-1}$ (A_0 and A_n for symmetry of notation are written for A and B). In order to be specific, we assume that these points are numbered in the direction from A to B, although it would in no way hinder us to number them in the opposite direction.

† A continuous curve is called simple, if it is given parametrically, and each of its points is obtained only from one value of the parameter; in the case of a closed simple curve an exception is the unique point at which "the curve closes" and which corresponds to the two extreme values of the parameter.

Taking any point M_i on the arc A_iA_{i+1} of the curve, we will evaluate the density $\varrho(M_i)$ at this point. Assuming that the density is approximately the same at all points of this section, and denoting the length of the arc A_iA_{i+1} by σ_i, for the mass m_i of this arc we have the approximation

$$m_i = \varrho(M_i)\sigma_i,$$

and for the whole mass which we are seeking we have the expression

$$m \doteq \sum_{i=0}^{n-1} \varrho(M_i)\sigma_i.$$

The error in this, resulting from the approximation made above will tend to zero, if the lengths σ_i of all the sections tend to zero.

Thus, denoting by λ the greatest of the lengths σ_i, to obtain an accurate formula it remains only to take limits:

$$m = \lim_{\lambda \to 0} \sum_{i=0}^{n-1} \varrho(M_i)\sigma_i.$$

We will begin by studying, in general, limits of this kind; abstracting from the problem under consideration, we will take an arbitrary "function of a point" $f(M) = f(x, y)$, given along a continuous plane rectifiable curve (K)†, and repeat the given process: splitting the curve (K) into the elementary arcs A_iA_{i+1} and choosing arbitrary points $M_i(\xi_i, \eta_i)$ on them, we calculate the values $f(M_i) = f(\xi_i, \eta_i)$ at these points and form the sum

$$\sum_{i=0}^{n-1} f(M_i)\sigma_i = \sum_{i=0}^{n-1} f(\xi_i, \eta_i)\sigma_i;$$

it is also a kind of integral sum.

The finite limit of this sum as $\lambda = \max \sigma_i$ *tends to zero is called the* curvilinear integral (of the first kind‡) *of the function* $f(M)$

† Here it is presupposed that at the beginning there is given some rectilinear system of coordinates.

‡ As distinct from curvilinear integrals of the second kind, considered below [Sec. 330].

$= f(x, y)$, *taken on the curve or path* (K), and it is denoted by the symbol

$$\int_{(K)} f(M)ds = \int_{(K)} f(x, y)ds \tag{1}$$

(where s is the length of the arc of the curve, and ds recalls the elementary arcs σ_i).

The precise characterization of the preceding process can be left to the reader.

Thus, the expression obtained above for the mass of an actual curve can be written thus:

$$m = \int_{(K)} \varrho ds. \tag{2}$$

We note in particular that in the given definition no part was played by the direction in which the path (K) *was given.* If, for instance, this curve is not closed and by (AB) and (BA) we mean the differently directed curves, then

$$\int_{(AB)} f(M)ds = \int_{(BA)} f(M)ds.$$

By analogous reasoning, we could introduce the concept of an integral extended over the space curve (K):

$$\int_{(K)} f(M)ds = \int_{(K)} f(x, y, z)ds\dagger.$$

In view of the fact that there are no instances which are new in principle, there is no need to give this here in detail.

328. The reduction to an ordinary definite integral. We presuppose that on the curve (K) there is an arbitrarily established direction (one of two possibilities), so that the position of the point M on the curve can be defined by the length of the arc $s = \widetilde{AM}$, measured from the initial point A. Then the curve (K) is expressed parametrically by equations of the form:

$$x = x(s), \quad y = y(s) \quad (0 \leqslant s \leqslant S),$$

and the function $f(x, y)$, given at points of the curve, will reduce to the compound function $f(x(s), y(s))$ of the variable s.

---

† At the beginning there is given some rectilinear system of coordinates. The function f is defined only at the points of the curve (K).

If by $s_i (i = 0, 1, \ldots, n)$ we denote the values of the arcs corresponding to the dividing points A_i chosen on the curve AB, then, clearly, $\sigma_i = s_{i+1} - s_i = \Delta s_i$. Denoting by $\bar{s}_i$ the value of s defining the point M_i (where, clearly, $s_i \leqslant \bar{s}_i \leqslant s_{i+1}$), we see that the integral sum for the curvilinear integral

$$\sum_{i=0}^{n-1} f(M_i)\,\sigma_i = \sum_{i=0}^{n-1} f\big(x(\bar{s}_i), y(\bar{s}_i)\big)\,\Delta s_i$$

is at the same time the integral sum for the usual definite integral, so that we have at once:

$$\int_{(K)} f(M)\,ds = (\mathrm{R}) \int_0^S f\big(x(s), y(s)\big)\,ds^{\dagger}, \tag{3}$$

where the existence of one of the integrals implies the existence of the other.

The integral clearly exists, for instance, in the case when the function $f(M)$ is continuous‡, which we will presuppose from now on.

Now let the curve (K) be given by the arbitrary parametric equations

$$x = \varphi(t), \quad y = \psi(t) \quad (t_0 \leqslant t \leqslant T),$$

where the functions φ and ψ are continuous together with their derivatives φ' and ψ'; we presuppose, moreover, that there are no multiple points on the curve. Then the curve is known to be rectifiable, and if an increase in the arc $s = \overset{\frown}{AM} = s(t)$ corresponds to an increase in the parameter t,

$$s'(t) = \sqrt{\{[\varphi'(t)]^2 + [\psi'(t)]^2\}}$$

† The sign (R) means that the integral is here understood to be in accordance with the usual, Riemann, definition.

‡ We have in view continuity at the points of the curve (K) and along it. In the "ε–δ" terminology, this means that for $\varepsilon > 0$ we can find a $\delta > 0$ such that $|f(M') - f(M)| < \varepsilon$ where $\overset{\frown}{MM'} < \delta$ (M and M' are points of the curve). With this hypothesis the compound function $f(x(s), y(s))$ is also a continuous function of s in so far as $x(s)$ and $y(s)$ are continuous.

[Secs. 201, 202]. Changing the variable in the integral in (3) on the right, we obtain at once:

$$\int_{(K)} f(M)\,ds = (\mathrm{R})\int_{t_0}^{T} f(\varphi(t), \psi(t))\ \sqrt{\{[\varphi'(t)]^2 + [\psi'(t)]^2\}}\,dt. \qquad (4)$$

Thus, *to evaluate a curvilinear integral of the first kind it is necessary to replace the variables x and y in the function under the integral by the expressions for the coordinates in terms of the parameter, and the factor ds by the differential of the arc as a function of the parameter.*

If as t increases, the arc AM were to decrease, then we would only need to go over to the arc BM in order to arrive at the formula (4) again. Thus, in this formula the lower limit of the integral on the right must be less than the upper limit, whatever may be the parameter given for the curve.

In the case of a curve given by the explicit equation:

$$y = y(x) \qquad (a \leqslant x \leqslant b),$$

formula (4) takes the form:

$$\int_{(K)} f(M)\,ds = (\mathrm{R})\int_{a}^{b} f(x,\ y(x))\ \sqrt{\{1 + [y'(x)]^2\}}\,dx. \qquad (5)$$

We can also change this relation to the other form. By the hypothesis of the continuity of the function $y(x)$ together with its derivative $y'(x)$ the curve (K) at every point will have a definite tangent, not parallel to the axis of y. Denoting by α the angle the tangent makes with the axis of x, we obtain:

$$\tan\alpha = y'(x) \qquad |\cos\alpha| = \frac{1}{\sqrt{\{1 + [y'(x)]^2\}}}.$$

Therefore

$$\int_{(K)} f(M)\,ds = \int_{a}^{b} \frac{f(x, y(x))}{|\cos\alpha|}\,dx. \qquad (6)$$

In particular, because, clearly,

$$\int_{(K)} ds = S,$$

where S denotes the length of the whole curve (K), then

$$S = \int_{a}^{b} \frac{dx}{|\cos\alpha|}. \qquad (7)$$

Remark. We obtained formula (7) as a result of formal transformations. If we were to define the length of the arc of the curve as the limit of the perimeter of the circumscribed (and not inscribed) broken lines, then this definition—in the case of an explicitly given curve—would apply immediately to formula (7). We leave it to the reader to verify this.

329. Examples. *(1)* We will calculate the integral $I = \int\limits_{(K)} xy\,ds$, if (K) is the quarter of the ellipse $x^2/a^2 + y^2/b^2 = 1$ lying in the first quadrant.

Starting from the parametric representation of the ellipse: $x = a\cos t$, $y = b\sin t$, we will have:

$$x_t' = -a\sin t, \quad y_t' = b\cos t, \quad \sqrt{(x_t'^2 + y_t'^2)} = \sqrt{(a^2\sin^2 t + b^2\cos^2 t)}.$$

The calculation can be performed according to formula (4):

$$I = \int\limits_0^{\pi/2} a\cos t \cdot b\sin t \cdot \sqrt{(a^2\sin^2 t + b^2\cos^2 t)}\,dt$$

$$= \frac{ab}{2}\int\limits_0^{\pi/2} \sin 2t \sqrt{\left(a^2\,\frac{1-\cos 2t}{2} + b^2\,\frac{1+\cos 2t}{2}\right)}\,dt.$$

Here we set $\cos 2t = z$, then $\sin 2t\,dt = -(1/2)dz$ and

$$I = \frac{ab}{4}\int\limits_{-1}^{1} \sqrt{\left(\frac{a^2+b^2}{2} + \frac{b^2-a^2}{2}\,z\right)}\,dz$$

$$= \frac{ab}{4}\,\frac{2}{b^2-a^2}\,\frac{2}{3}\left[\frac{a^2+b^2}{2} + \frac{b^2-a^2}{2}\,z\right]^{3/2}\Bigg|_{-1}^{1} = \frac{ab}{3}\,\frac{a^2+ab+b^2}{a+b}.$$

Remark. For the majority of curves one meets (the ellipse, hyperbola, sine curve and so on) the length of the arc cannot be expressed in elementary functions, because ds is not integrable in a finite form. Nevertheless, the integral $\int\limits_{(K)} f(x, y)\,ds$ for such curves can sometimes also be calculated in elementary functions (as in the preceding example), because the adjunction of the factor $f(x, y)$ changes the whole structure of the differential under the integral.

Problems connected with masses distributed continuously along a material curve lead in a natural way to curvilinear integrals of the kind under consideration.

(2) We have already discussed in Chapter 12 [Sec. 206] the calculation of the moments of a plane curve about the coordinate axes, and also the coordinates of its centre of gravity with the hypothesis that its "linear density" $\varrho = 1$. The reader can easily extend the formulae obtained to the general case of continuously distributed masses. If we use the concept of a curvilinear integral which we have introduced, then the results can be written in the following form:

$$K_y = \int\limits_{(K)} \varrho x\,ds, \qquad K_x = \int\limits_{(K)} \varrho y\,ds,$$

$$x_c = \frac{K_y}{m} = \frac{\int\limits_{(K)} \varrho x\, ds}{\int\limits_{(K)} \varrho\, ds}, \qquad y_c = \frac{K_x}{m} = \frac{\int\limits_{(K)} \varrho y\, ds}{\int\limits_{(K)} \varrho\, ds}.$$

(3) We will indicate one further example of a curvilinear integral of the first kind—on the problem of the attraction of particles to a curve.

As is well known, by Newton's law, a particle M of mass m attracts a particle M_0 of mass m_0 with a force directed from M_0 to M and numerically equal to kmm_0/r^2, where r is the distance M_0M, and k is a coefficient depending on the choice of the fundamental unit of measurement; however, for simplicity we will usually assume that it is equal to one.

If the point M_0 is attracted by a system of points $M_1, M_2, \ldots, M_n$ with masses $m_1, m_2, \ldots, m_n$, then the resultant force, or resultant, is obtained by adding geometrically the forces of attraction of the separate points. At the same time the projections of the resultant force on the coordnate axes are equal to the algebraic sums of the projections of the separate forces.

If we denote the projections of the resultant on the axes by X and Y, and the angle made by the vector $\vec{r_i} = \overrightarrow{M_0M_i}$ with the x-axis by θ_i (Fig. 9), then, clearly,

$$X = \sum_{i=1}^{n} \frac{m_0 m_i}{r_i^2} \cos\theta_i, \qquad Y = \sum_{i=1}^{n} \frac{m_0 m_i}{r_i^2} \sin\theta_i,$$

where r_i, as usual, denotes the length of the vector $\vec{r_i}$.

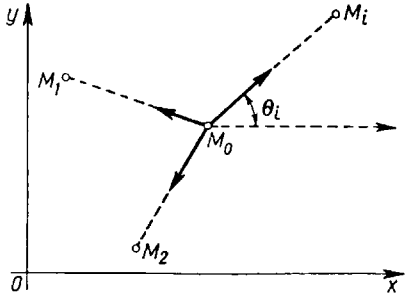

FIG. 9.

Now let the attracting masses be distributed continuously along the curve (K). To find the attraction we break the curve up into sections and, concentrating the mass of each section at a point M_i arbitrarily chosen on it, we will find the approximate values of the projections of the resultant on the axes:

$$X \doteq \sum_{i} \frac{m_0 \varrho(M_i)\sigma_i}{r_i^2} \cos\theta_i, \qquad Y \doteq \sum_{i} \frac{m_0 \varrho(M_i)\sigma_i}{r_i^2} \sin\theta_i,$$

because in this case the mass of a separate section is equal to $\varrho(M_i)\sigma_i$. If we make all the σ_i tend to zero, then as the limit we obtain exact equations, where the sums become integrals:

$$X = m_0 \int_{(K)} \frac{\varrho(M)\cos\theta}{r^2}\, ds, \qquad Y = m_0 \int_{(K)} \frac{\varrho(M)\sin\theta}{r^2}\, ds; \tag{8}$$

here r denotes the length of the vector $\vec{r} = \overrightarrow{M_0M}$, and θ the angle made by it with the x-axis.

We will find, for instance, the attraction of a homogeneous semicircle (when $\varrho = 1$) on a unit mass fixed at its centre.

We locate the origin of the coordinates at the centre of the semicircle and draw the x-axis along its edge (Fig. 10).

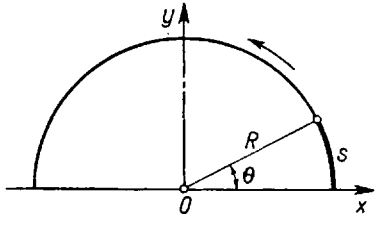

FIG. 10.

From considerations of symmetry $X = 0$, so we just have to find Y. By formula (8)

$$Y = \int \frac{\sin\theta}{r^2}\, ds.$$

But in our case $r = R$ (the radius of the semicircle) and $ds = R\, d\theta$. Therefore

$$Y = \frac{1}{R}\int_0^{\pi} \sin\theta\, d\theta = \frac{2}{R}.$$

§ 2. Curvilinear integrals of the second kind

330. The definition of curvilinear integrals of the second kind. Considering now the concept, which is more important in practice, of a curvilinear integral of the second kind; we give its definition immediately, postponing the applications of this concept to later sections [see, for instance, Sec. 335]. Let there be given a simple curve (AB) (which we presuppose, for the time being, not to be closed) and let there again be given a function $f(x, y)$, along

it†. Breaking up the curve at the points $A_i(x_i, y_i)$ into sections, we choose on the section A_iA_{i+1} of the curve an arbitrary point $M_i(\xi_i, \eta_i)$, and we evaluate at this point, as above, the value of the function $f(M_i) = f(\xi_i, \eta_i)$. But this time we multiply this value not by the length of the arc A_iA_{i+1}, but by the magnitude of the projection of this arc, say, on the x-axis, i.e. by $x_{i+1} - x_i = \Delta x_i$; then we form the integral sum

$$\sigma = \sum_{i=0}^{n-1} f(M_i)\Delta x_i = \sum_{i=0}^{n-1} f(\xi_i, \eta_i)\Delta x_i.$$

The finite limit I of this sum as $\mu = \max A_iA_{i+1}$ tends to zero is called the curvilinear integral (of the second kind) *of $f(M)dx$, taken on the curve or path (AB)* and it is denoted by

$$I = \int_{(AB)} f(M)dx = \int_{(AB)} f(x, y)dx. \tag{1}$$

Analogously, multiplying the value $f(M_i)$ not by Δx_i, but by Δy_i, i.e. by the projection of the arc A_iA_{i+1} on the y-axis, and forming the sum

$$\sigma^* = \sum_{i=0}^{n-1} f(M_i)\Delta y_i = \sum_{i=0}^{n-1} f(\xi_i, \eta_i)\Delta y_i$$

as its limit, we obtain the curvilinear integral (of the second kind) of $f(M)dy$:

$$I^* = \int_{(AB)} f(M)dy = \int_{(AB)} f(x, y)dy. \tag{2}$$

If two functions $P(M) = P(x, y)$ and $Q(M) = Q(x, y)$ are defined along the curve (AB), and the integrals

$$\int_{(AB)} P(M)dx = \int_{(AB)} P(x, y)dx, \quad \int_{(AB)} Q(M)dy = \int_{(AB)} Q(x, y)dy$$

exist, then their sum is called a curvilinear integral ("of the general form") and we set

$$\int_{(AB)} P(x, y)dx + Q(x, y)dy = \int_{(AB)} P(x, y)dx + \int_{(AB)} Q(x, y)dy‡. \tag{3}$$

† See the first footnote on p. 220.

‡ The reader will find an historical note about curvilinear integrals in the Remark in Sec. 350.

Now we will compare the definition of a curvilinear integral of the second kind (1) [or (2)] with the definition of a curvilinear integral of the first kind [see Sec. 327, (1)]. With their obvious similarity, the two definitions nevertheless have an essential difference, which we will emphasize yet again: in the case of the integral of the first kind in forming the integral sum the value of the function $f(M_i)$ *is multiplied by the length* $\sigma_i = \Delta s_i$ *of the section* A_iA_{i+1} *of the curve,* and in the case of the integral of the second kind this value of $f(M_i)$ *is multiplied by the projection* Δx_i *(or* Δy_i*) of the given section on the x-axis (or on the y-axis).*

We saw that the direction of the path (AB), along which the integration is taken, plays no part in the case of an integral of the first kind, because the length σ_i of the arc A_iA_{i+1} does not depend on this direction. The situation is different in the case of an integral of the second kind: the projection of the given arc on one or other of the axes depends essentially on the direction of the arc and changes sign when this direction changes. Thus, for integrals of the second kind it will be true that

$$\int_{(BA)} f(x, y)dx = -\int_{(AB)} f(x, y)dx$$

and analogously

$$\int_{(BA)} f(x, y)dy = -\int_{(AB)} f(x, y)dy,$$

where from the existence of the integrals on the right follows the existence of the integrals on the left, and conversely.

In a similar way we can introduce the concept of a curvilinear integral of the second kind extended over the space curve (say, one which is not closed) (AB). In fact, if the function $f(M) = f(x, y, z)$ is given at the points of this curve, then, as above, we construct the sum

$$\sigma = \sum_{i=0}^{n-1} f(\xi_i, \eta_i, \zeta_i)\,\Delta x_i$$

and consider the limit of this "integral" sum under the hypothesis that $\mu = \max \overline{A_iA_{i+1}}$ tends to zero. *This limit is called the* curvilinear

integral (of the second kind) *of* $f(M)dx$ and it is denoted by the symbol

$$\int_{(AB)} f(M)\,dx = \int_{(AB)} f(x,\ y,\ z)\,dx.$$

Analogously we define the integrals of the form

$$\int_{(AB)} f(M)\,dy = \int_{(AB)} f(x,\ y,\ z)\,dy$$

and

$$\int_{(AB)} f(M)\,dz = \int_{(AB)} f(x,\ y,\ z)\,dz.$$

Finally, we consider also the integral ("of the general form")

$$\int_{(AB)} P\,dx + Q\,dy + R\,dz = \int_{(AB)} P\,dx + \int_{(AB)} Q\,dy + \int_{(AB)} R\,dz.$$

Again a change in the direction of the integration changes the sign of the integral.

We note in conclusion that the simplest properties of an integral defined in the usual way carry over easily to the curvilinear integrals under consideration; we will not pause to discuss this.

331. The existence and evaluation of a curvilinear integral of the second kind. Let the curve $(K) = (AB)$ be given by the parametric equations

$$x = \varphi(t), \quad y = \psi(t), \tag{4}$$

where the functions φ and ψ are continuous, and as the parameter t changes from α to β the curve is described in the direction from A to B. We will presuppose that the function $f(x, y)$ along the curve (AB) is continuous†.

If we are discussing the integral (1), then in addition we stipulate the existence and continuity of the derivative $\varphi'(t)$.

With these hypotheses the curvilinear integral (1) exists, and the equation

$$\int_{(AB)} f(x,\ y)\,dx = (\mathrm{R}) \int_{\alpha}^{\beta} f(\varphi(t),\ \psi(t))\varphi'(t)\,dt \tag{5}$$

† Here we might make a remark analogous to the second footnote on p. 222, but replacing the arc $\overset{\frown}{MM'}$ by the chord MM'.

holds. Thus, *to evaluate the curvilinear integral* (1) *it is necessary to replace the variables x and y in the function under the integral by their expressions* (4) *in terms of a parameter, and the factor dx by the differential of the variable x as a function of the parameter.* The order in which the limits are placed in the last integral corresponds this time to the choice of the direction along the curve.

Now we give the proof. Let the points A_i $(i = 0, 1, 2, \ldots, n)$, taken on the curve, be defined by the values t_i of the parameter, and the point M_i chosen on the arc A_iA_{i+1} by the value τ_i (clearly, this lies between t_i and t_{i+1}). Then the integral sum

$$\sigma = \sum_{i=0}^{n-1} f(\xi_i, \eta_i)\Delta x_i,$$

if we take into account that

$$\Delta x_i = \varphi(t_{i+1}) - \varphi(t_i) = \int_{t_i}^{t_{i+1}} \varphi'(t)\,dt,$$

can be written in the form

$$\sigma = \sum_{i=0}^{n-1} f(\varphi(\tau_i), \psi(\tau_i)) \int_{t_i}^{t_{i+1}} \varphi'(t)\,dt.$$

On the other hand, the integral on the right in (5)† can also be written in the form of a sum:

$$I = \int_{\alpha}^{\beta} f(\varphi(t), \psi(t))\varphi'(t)\,dt = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(\varphi(t), \psi(t))\varphi'(t)\,dt.$$

Hence

$$\sigma - I = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} [f(\varphi(\tau_i), \psi(\tau_i)) - f(\varphi(t), \psi(t))]\varphi'(t)\,dt.$$

Given an arbitrary $\varepsilon > 0$, we presuppose now that all the Δt_i are small enough that in the intervals $[t_i, t_{i+1}]$ the oscillations of the

† The actual existence of the integral is obvious in view of the continuity of the function under the integral.

continuous function $f(\varphi(t), \psi(t))$ will be less than ε. Because the continuous function $\varphi'(t)$ is bounded: $|\varphi'(t)| \leqslant L$, then we have

$$|\sigma - I| < \varepsilon L |\beta - \alpha|.$$

Thus, as the value of $\lambda = \max \Delta t_i$ tends to 0†

$$\lim \sigma_i = I,$$

which simultaneously proves both the existence of the curvilinear integral and the required equation.

Turning to the integral (2), we can establish its existence in a similar way and prove the formula

$$\int_{(AB)} f(x, y)\, dy = (\mathrm{R}) \int_{\alpha}^{\beta} f(\varphi(t), \psi(t))\psi'(t)\, dt \qquad (5^*)$$

with the hypothesis that the continuous derivative $\psi'(t)$ exists.

Finally, if we are discussing the integral of the general form

$$\int_{(AB)} P(x, y)\, dx + Q(x, y)\, dy,$$

where P and Q are continuous functions, then we impose the requirement on the curve (AB) that both the functions (4) have continuous derivatives. With this hypothesis the formula

$$\int_{(AB)} P(x, y)\, dx + Q(x, y)\, dy = (\mathrm{R}) \int_{\alpha}^{\beta} [P(\varphi(t), \psi(t))\varphi'(t) + Q(\varphi(t), \psi(t))\psi'(t)]\, dt \qquad (6)$$

will be valid.

The definition of the curvilinear integral and the method given here of reducing it to the usual definition of an integral can immediately be extended to the case of the curve (4), which intersects itself, provided only that the direction along it is defined, as before as the direction in which the parameter t increases monotonically from α to β.

In conclusion, we mention some cases when the evaluation of the curvilinear integral is particularly simple.

† This is equivalent—in the case of a non-closed curve—to the tending to zero of the largest of the chords [Sec. 200].

Let the integral (2) be taken on the curve given by the explicit equation:

$$y = y(x),$$

where the variation of the point from A to B corresponds to the change of x from a to b.

Then, without any hypotheses about the curve, except its continuity, we have

$$\int_{(AB)} f(x, y)\,dx = (\mathrm{R})\int_a^b f(x, y(x))\,dx. \tag{7}$$

Analogously, if the integral (2*) is extended over a continuous curve, given by an explicit equation, but of the other type:

$$x = x(y),$$

where y varies from c to d, then

$$\int_{(AB)} f(x, y)\,dy = (\mathrm{R})\int_c^d f(x(y), y)\,dy. \tag{7*}$$

Finally, if the integral (2) is extended over the rectilinear section (AB), parallel to the y-axis, then it is equal to 0 (because in this case all the Δx_i are zero, and with them so is their sum σ). Analogously the integral (2*) is also equal to 0; it is taken along the rectilinear section, parallel to the x-axis.

Remark. If the curve (K) is broken up into a finite number of curves which join one another and along each of which the curvilinear integral exists and can be evaluated by one of the given formulae, then it is easy to calculate the integral along the whole curve (K), as the sum of the integrals along its parts.

332. The case of a closed contour. The orientation of the plane. We turn to the consideration of the closed contour (K), i.e. the case when the beginning A and the end B of the path of the integration coincide. Taking a point C on the curve distinct from A, and taking into account the choice of the direction along the curve (in Fig. 11 it is indicated by an arrow), we let by definition:

$$\int_{(K)} = \int_{AMC} + \int_{CNA},$$

with the hypothesis that the integrals on the right exist.

It is easy to prove that the existence and value of the integral do not depend on the choice of the points A and C. Moreover, for the closed contour (K) the formulae (5), (5*), (6), derived in the preceding section, are applicable.

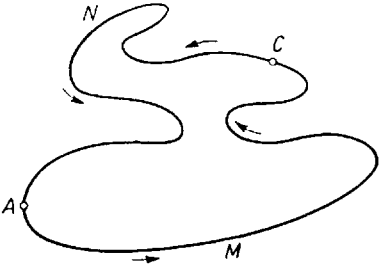

FIG. 11.

The peculiarity of the case under consideration lies in the fact that the given beginning and end-points (which coincide) this time do not determine the direction in which the curve (K) is drawn. We could specially indicate in each case which direction we have in mind. It is appropriate to do this if we are discussing a space curve. In the case of a plane closed contour (K) we usually do something else.

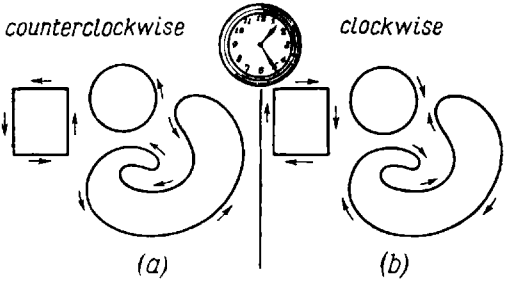

FIG. 12.

Of the two possible directions of rotation for a given plane—"counterclockwise" and "clockwise"—one is chosen as positive: this gives a definite *orientation of the plane*. If we regard the counterclockwise rotation as positive, then the orientation of the plane is said to be right, and in the other case left.

In the case of the right orientation of the plane we in fact take counterclockwise rotation as the basis of the definition of the positive direction along a simple closed contour (Fig. 12*a*). In fact, this definition is sufficiently clear only for contours which are almost circumferences. Therefore let us be more precise: the *positive direction* of the circuit of a simple closed curve is that for which the part of the domain bounded by the contour, which is nearest to an observer, turns out to lie on the left of the observer (Fig. 12*a*). In the case of a left oriented plane the positive circuit of the contour will be the clockwise one, so that the domain will remain on the right of the observer (Fig. 12*b*).

We note that the ordering of the coordinate axes on the plane is always connected with its orientation: the y-axis is obtained from the x-axis by rotating it through an angle of $\pi/2$ counterclockwise for a right orientation of the plane and clockwise for a left orientation (Figs. 13*a*, *b*). In the first case the coordinate system itself is called right, and in the second left.

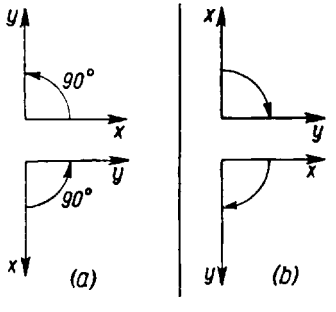

FIG. 13.

After these explanations we shall assume that *if the path of integration* (K) *is a simple closed curve, then by the symbol*

$$\oint_{(K)} P\,dx + Q\,dy$$

where the direction of the circuit of the contour is not given, *we understand an integral taken in the positive direction.* Of course, this

convention does not prevent our considering, where necessary, an integral taken in the negative direction, but we will denote it by

$$-\int_{(K)} P\,dx + Q\,dy.$$

333. Examples. *(1)* We will calculate the value of the curvilinear integral

$$H = \int_{(L)} 2xy\,dx + x^2\,dy,$$

taken along the path (L), joining the points O $(0, 0)$ and A $(1, 1)$, if the path (L) is: (a) the straight line $y = x$, (b) the parabola $y = x^2$, (c) the parabola $x = y^2$, (d) the cubic parabola $y = x^3$ (Fig. 14).

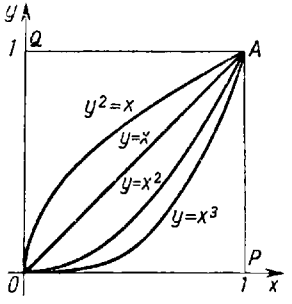

FIG. 14.

(a) Because $dy = dx$, then

$$\int_{(L)} 2xy\,dx + x^2\,dy = \int_0^1 3x^2\,dx = 1;$$

(b) $dy = 2x\,dx$, $H = \int_0^1 4x^3\,dx = 1$;

(c) $dx = 2y\,dy$, $H = \int_0^1 5y^4\,dy = 1$;

(d) $dy = 3x^2\,dx$, $H = \int_0^1 5x^4\,dx = 1$.

(2) To calculate the curvilinear integral

$$G = \int_{(L)} xy\,dx + (y - x)\,dy$$

with the same paths of integration.

Answer. (a) 1/3, (b) 1/12, (c) 17/30, (d) $-1/20$.

(3) To find the curvilinear integral

$$I = \int\limits_{(OA)} (x - y^2)\,dx + 2xy\,dy,$$

if as the path of integration one takes one of the following lines joining $O\,(0, 0)$ and $A\,(1, 1)$ (Fig. 14): (a) the rectilinear segment $OA (y = x)$; (b) the broken contour OPA, consisting of the segment OP of the x-axis ($y = 0$) and the segment PA of the straight line $x = 1$; (c) the broken contour OQA, consisting of the segment OQ of the y-axis ($x = 0$) and the segment QA of the straight line $y = 1$.

(a) $y = x$, so $dy = dx$, and

$$I = \int_0^1 (x + x^2)\,dx = \frac{5}{6}.$$

(b) In this case it is natural to break up the path of integration into two segments:

$$I = \int\limits_{(OPA)} = \int\limits_{(OP)} + \int\limits_{(PA)} = I_1 + I_2.$$

Along OP we have: $y = 0$ and $dy = 0$, so that

$$I_1 = \int_0^1 x\,dx = \frac{1}{2}.$$

Along PA it will be: $x = 1$ and $dx = 0$, therefore

$$I_2 = \int_0^1 2y\,dy = 1.$$

Thus, finally, $I = 3/2$.

(c) Like the preceding case, we find (because the integral along the segment OQ is equal to zero):

$$I = \int\limits_{(QA)} = \int_0^1 (x - 1)\,dx = -\frac{1}{2}.$$

(4) The same holds for the integral

$$I = \int\limits_{(OA)} (y^2 + 2xy)\,dx + (2xy + x^2)\,dy.$$

Answer. In all cases $I = 2$.

Remark. The reader has probably already noticed the differences between the results of examples *(1)* and *(4)*, on the one hand, and *(2)* and *(3)* on the other. The values of the integrals considered in *(1)* and *(4)* turn out to be independent of the lines joining the first and last points. On the contrary, in examples *(2)* and *(3)* we encountered integrals whose values depend on which of the

lines joins the first and last points. In the following chapter [§ 4] we will deal with this problem in particular and explain its importance.

(5) We will evaluate the integral

$$K = \int_{(L)} y^2\,dx - x^2\,dy,$$

where (L) is a circle of radius 1 with centre: (a) at the origin of the coordinates or (b) at the point $(1, 1)$.

(a) Starting from the parametric equations $x = \cos t$, $y = \sin t$, where t varies from 0 to 2π, by formula (6) we will have

$$K = -\int_0^{2\pi} (\sin^3 t + \cos^3 t)\,dt = 0.$$

(b) Analogously with the help of the parametric representations

$$x - 1 = \cos t, \quad y - 1 = \sin t$$

we obtain

$$K = -\int_0^{2\pi} (2 + \sin t + \cos t + \sin^3 t + \cos^3 t)\,dt = -4\pi.$$

334. The connection between curvilinear integrals of both kinds. We will consider the simple smooth† curve $(K) \equiv (AB)$ and, choosing as a parameter the arc $s = \overset{\frown}{AM}$, we represent it by means of the equations [Secs. 201, 202]

$$x = x(s), \quad y = y(s) \quad (0 \leqslant s \leqslant S);$$

the functions $x(s), y(s)$ will have continuous derivatives $x'(s), y'(s)$. If by α we denote the angle made with the x-axis by the tangent in the direction in which the arc increases, then, as we know [Sec. 211, (14)],

$$\cos\alpha = x'(s), \ \sin\alpha = y'(s).$$

If along the curve (K) there is given a continuous function $f(M) = f(x, y)$, then we have successively

$$\int_{(K)} f(M)\,dx = \int_0^S f(x(s),\ y(s))x'(s)\,ds$$

$$= \int_0^S f(x(s), y(s))\cos\alpha\,ds = \int_{(K)} f(M)\cos\alpha\,ds,$$

† The curve (4) is called smooth, if the functions φ and ψ have continuous derivatives, which do not vanish simultaneously.

and the curvilinear integral of the second kind is reduced to a curvilinear integral of the first kind.

Analogously we obtain

$$\int_{(K)} f(M)\,dy = \int_{(K)} f(M)\sin\alpha\,ds.$$

If, however, there are given two continuous curves $P(M) = P(x, y)$ and $Q(M) = Q(x, y)$ along the curve (K), then

$$\int_{(K)} P\,dx + Q\,dy = \int_{(K)} (P\cos\alpha + Q\sin\alpha)\,ds. \tag{8}$$

We emphasize that in all these formulae the angle α is taken with the direction of the tangent which corresponds to the direction of the curve (K) itself. If we change the direction of the curve, then not only the integral on the left changes, but also, arising from the change in direction of the tangent the angle α is changed by $\pm\pi$, so the integral on the right also changes sign.

Clearly, the given formulae remain valid also for a simple piecewise smooth curve†; it is easy to verify this if we write them down for each of the smooth pieces of the curve and add them together term by term.

Analogous reasoning can also be carried out for curvilinear integrals along a space curve. As a result we obtain the formula

$$\int_{(K)} P\,dx + Q\,dy + R\,dz = \int_{(K)} (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,ds, \tag{9}$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the tangent, with the hypothesis that its direction corresponds to the direction of the path of integration.

335. Applications to physical problems. We turn in conclusion to two important problems from the field of mechanics and physics, from which the reader will be able to form a picture of the part played by the concepts introduced in this section.

(1) The work of a field of force. At every point M of the plane xy (or a definite part of it) let there act on a unit mass embedded in it a definite force $\vec{F}$, *whose magnitude and direction depend only on the position of the point* M; if the mass m of the material particle embedded at M differs from unity, then the force

† This is what we call a curve formed from several smooth curves which join onto one another.

acting on it will be equal to $\overrightarrow{mF}$. Then the plane (or the part of it under consideration) is called *a (plane) field of force*, and the force $\vec{F}$, acting on the unit mass, is called *the tension of the field*. The effect of the force $\vec{F}$ is equivalent in magnitude and direction to the effect of its projections X, Y on the axes, which are clearly functions of the coordinates x, y of the point M:

$$X = X(x, y), \quad Y = Y(x, y).$$

If φ is the angle made by the vector $\vec{F}$ with the x-axis, then (Fig. 15)

$$X = F\cos\varphi, \quad Y = F\sin\varphi. \tag{10}$$

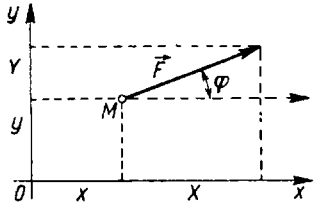

FIG. 15.

Now we presuppose that the material point $M(x, y)$, with unit mass, situated in a field, moves and describes some continuous curve (K) *in a definite direction.* The problem lies in calculating the work A which is done by the force of the field during this movement.

If the force acting on the particle kept a constant magnitude F and a constant direction, and the movement itself of the point were rectilinear, then, as we know, the work A could be expressed as the product of the distance l moved by the projection of the force on the direction of the motion:

$$A = Fl\cos\theta,$$

where θ is the angle between the force $\vec{F}$ and the direction of the motion.

In the case of non-rectilinear motion and a non-constant force the work is defined by a certain limit process †. We begin by defining the position of the point M on the curve (K) (Fig. 16) in terms of the length s of the arc $\overset{\frown}{AM}$. We consider an infinitely small element $MN = ds$ of the curve; we will assume that it is rectilinear and that the force $\vec{F}$ and the angle θ keep the same values approximately as the particle moves along ds. Then the corresponding element of work will be $dA = F\cos\theta\, ds$. Now it remains only to "sum" these elements along the whole curve (K), as a result of which the work A is expressed as a curvilinear integral of the first type:

$$A = \int_{(K)} F\cos\theta\, ds. \tag{11}$$

† However, here and later we prefer the shorter expression, which was introduced in the first volume [Sec. 204]. Passage to a limit is implied.

We introduce the angle α between the direction of the element ds (i.e. the direction of the tangent to the curve at the point M) and the x-axis. Clearly, $\theta = \varphi - \alpha$, so that

$$\cos\theta = \cos\varphi \cdot \cos\alpha + \sin\varphi \cdot \sin\alpha,$$

and the element of the integral, in view of (10), can be written:

$$(X\cos\alpha + Y\sin\alpha)ds.$$

The expression (11) for the work itself takes the form:

$$A = \int_{(K)} (X\cos\alpha + Y\sin\alpha)ds.$$

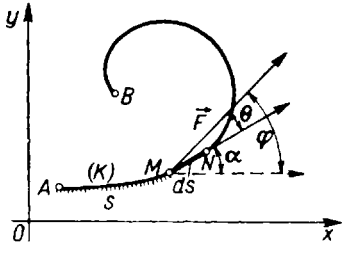

FIG. 16.

If we now take formula (8) into account, which establishes a connection between the curvilinear integrals of both kinds, then, finally, the work of the field of force can be expressed by a curvilinear integral of the second kind:

$$A = \int_{(K)} Xdx + Ydy. \tag{12}$$

This is the most usual expression for the work A; it is convenient for the investigation of a series of important problems connected with it: the work done depends either on a form of trajectory joining two given points; or the work along a closed trajectory will always be equal to zero [concerning this see below, Secs. 348–351].

(2) The motion of an incompressible fluid restricted to a plane. Such motion is characterized by the fact that, first, all particles lying on one vertical to some plane have the same velocity, so that to characterize the whole motion it is sufficient to consider the motion on only one plane†, and, secondly, the velocity $\vec{c}$ of a particle of the fluid depends only on the position of the particle, but not on the time. Thus, with every geometric point of the plane under consideration (or part of it) is associated a velocity of definite magnitude and direction; in other words, there is given some "field of velocity".

† Which we will choose as the plane xy.

If we denote the angle made by the vector $\vec{c}$ with the x-axis by φ, and the projections of this vector on the coordinate axes by u and v, then (Fig. 17*a*)

$$u = c_x = c\cos\varphi, \quad v = c_y = c\sin\varphi.$$

Now let us take some curve (K) in the plane xy and let us try to define the quantity Q of the fluid which flows along it, on a definite side of it, in unit time. Presupposing the fluid to be incompressible, the quantity of the fluid can be measured by the area covered by its figure. If in fact the fluid flows on the side opposite the one chosen, then the quantity of the fluid flowing past will be considered negative.

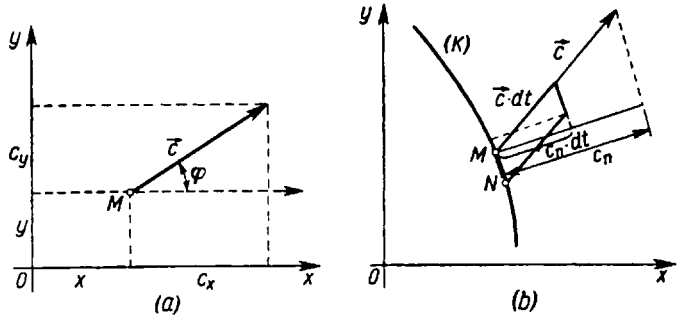

FIG. 17.

We will consider the element $MN = ds$ of the curve (K). In the time dt, past this element will flow a quantity of fluid equal to

$$c_n\,ds\,dt,$$

where c_n is the projection of velocity $\vec{c}$ on the normal n to the element ds, directed towards the chosen side of the curve. In fact, this quantity is equal to the area of the parallelogram with sides ds and dt, whose altitude is exactly the product $c_n\,dt$ (Fig. 17*b*)†. To calculate the quantity of the fluid flowing past the element ds in unit time, we sum the expression (13) with respect to the elements dt, which gives $c_n\,ds$. Summing the expressions we have found with respect to all the elements of the curve (K), we can represent the desired quantity Q of the fluid in the form of a curvilinear integral of the first kind

$$Q = \int_{(K)} c_n\,ds.$$

If, as always, α denotes the angle between the tangent and the x-axis, then $\lambda = \alpha + \pi/2$ gives the angle of the normal with the same axis. Then the angle

† Here we take the element ds to be rectilinear and assume that the velocities of the particles of the fluid at all points of it are identical and during the element of time dt keep the same magnitude and direction.

between the direction of the velocity $\vec{c}$ and the normal will be $\varphi - \lambda = \varphi - \alpha - \pi/2$, and

$$c_n = c\cos\left(\varphi - \alpha - \frac{\pi}{2}\right) = c\sin(\varphi - \alpha)$$

$$= c\sin\varphi\cos\alpha - c\cos\varphi\sin\alpha = v\cos\alpha - u\sin\alpha,$$

so that

$$Q = \int_{(K)} (v\cos\alpha - u\sin\alpha)ds.$$

Again taking into account formula (8), we arrive at the curvilinear integral of the second kind

$$Q = \int_{(K)} v dx - u dy. \tag{14}$$

Here it is important to emphasize that the direction of this curve must be taken so that the angle between the corresponding direction of the tangent and the direction of the normal chosen beforehand is equal to $+\pi/2$, because it was from precisely this hypothesis that formula (14) was derived.

If (K) is a closed contour and the integral (14) is regarded as being taken (as usual, Sec. 332) in the positive direction, then, in order that the given condition should be satisfied, the normal must be directed towards the interior of the domain bounded by the contour (K). Hence, in this case formula (14) gives the quantity of fluid flowing past the contour (K) in unit time on the inside of the domain. If we wish to obtain the quantity of fluid flowing on the outside of the domain bounded by the contour (K), then we need only change the sign in formula (14).

Further, if the field has neither "sources" nor "outlets" of fluid, then in any bounded domain the quantity of fluid remains constant. Therefore, whatever closed curve is taken, the integral (14), taken on it, must be equal to zero.

Thus, *if u and v are the component velocities of an incompressible fluid in a fixed plane, then when there are no sources or outlets*

$$\int_{(K)} v dx - u dy = 0$$

for any closed contour (K).

Later [Sec. 351, *(2)*] we will see that this result, obtained by means of physical considerations, enables us to give a certain analytical characterization of the functions u and v.

CHAPTER 21

DOUBLE INTEGRALS

§ 1. The definition and simplest properties of double integrals

336. The problem of the volume of a cylindrical body. In the same way as the problem of the area of a curvilinear trapezoid led us to the concept of a simple definite integral [Sec. 175], the analogous problem of the volume of a cylindrical body leads us to a new concept—a double (definite) integral.

We consider the body (V), which is bounded from above by the surface

$$z = f(x, y), \tag{1}$$

with sides formed by a cylindrical surface with generators parallel to the z-axis, and is, finally, bounded from below by the plane figure (P) on the xy plane (Fig. 18). It is required to find the volume V of the body.

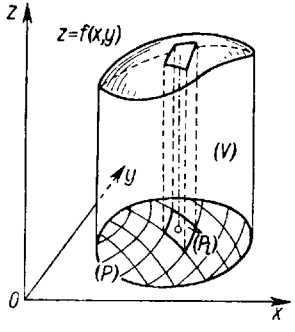

FIG. 18.

To solve this problem we have recourse to the usual method of the integral calculus, which consists of breaking up the quantity we are seeking into elementary parts, calculating each part approximately, summing and consequently taking limits. For this purpose we break

up the domain (P) by means of a net of curves into the parts $(P_1), (P_2), \ldots, (P_n)$ and we consider a series of cylindrical columns, which have as their bases these parts of the domain and together form the given body.

To calculate the volume of the separate columns we shall take in each figure (P_i) an arbitrary point: (ξ_i, η_i). If we take each column to be approximately a right cylinder with altitude equal to the z-coordinate, $f(\xi_i, \eta_i)$, then the volume of a separate column is approximately equal to

$$f(\xi_i, \eta_i) \cdot P_i,$$

where P_i denotes the area of the figure (P_i). In this case the approximate expression for the volume of the whole body will be:

$$V \doteq \sum_{i=1}^{n} f(\xi_i, \eta_i) P_i.$$

To make this equation more precise we will decrease the size of the areas (P_i), while increasing their number. As a limit, when the greatest in diameter of all the domains (P_i) tends to zero, this equation will become exact, so that

$$V = \lim \sum_{i=1}^{n} f(\xi_i, \eta_i) P_i,$$

and the required problem is solved.

A limit of this form is also *the double integral of the function* $f(x, y)$ *over the domain* (P); it is denoted by the symbol

$$\iint_{(P)} f(x, y)\, dP.$$

The formula for the volume obtained above takes the form

$$V = \iint_{(P)} f(x, y)\, dP^{\dagger}. \tag{2}$$

Thus, a double integral is a direct generalization of the concept of a simple definite integral in the case of a function of two variables.

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† In the case of a continuous surface (1) and a quadrable figure (P) the derivation of this formula can, without difficulty, be given a completely rigorous form; see the remark in Sec. 340.

It also plays an important part in the defining of different geometrical and physical quantities.

337. The reduction of a double integral to a repeated integral. Continuing to treat a double integral geometrically as the volume of a cylindrical body, we will give here an indication of how it is calculated by means of reducing it to a repeated integral.

In Vol. I we have already discussed the problem of calculating the volume of a body (V) by means of its transverse sections [Sect. 198]. We recall the formula referred to there. Let the body be bounded by the planes $x = a$ and $x = b$ (Fig. 19). We suppose that the section

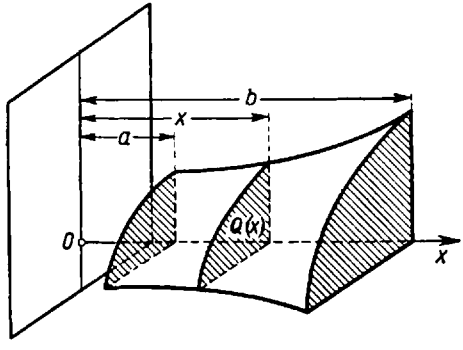

FIG. 19.

of the body on the plane perpendicular to the x-axis and corresponding to the abscissa $x (a \leqslant x \leqslant b)$ has area $Q(x)$. Then the volume of the body, under the hypothesis that it exists, is expressed by the formula

$$V = \int_a^b Q(x)\,dx. \tag{3}$$

Now we apply this formula to calculating the volume of the cylindrical body, which was discussed in the preceding section. We begin with a simple case—when the base of the body is a rectangle $[a, b; c, d]$ (Fig. 20).

The section of the body on the plane $x = x_0 (a \leqslant x_0 \leqslant b)$ is a curvilinear trapezoid $\alpha\beta\delta\gamma$. To find its area we project this figure

on the yz plane; we obtain the trapezoid $\alpha_1\beta_1\delta_1\gamma_1$ congruent with it, because the projection is performed without distortion. Thus,

$$Q(x_0) = \text{area}\quad \alpha\beta\delta\gamma = \text{area}\quad \alpha_1\beta_1\delta_1\gamma_1.$$

But the equation of the line $\gamma_1\delta_1$ on the yz plane, clearly, will be

$$z = f(x_0, y) \qquad (c \leqslant y \leqslant d).$$

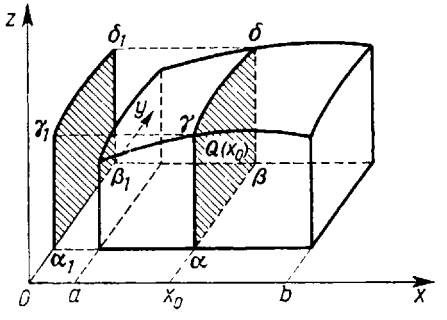

FIG. 20.

Using the well-known expression for the area of a curvilinear trapezoid in the form of a definite integral we will have

$$Q(x_0) = \int_c^d f(x_0, y)\,dy.$$

Because our reasoning referred to any section, then in general for $a \leqslant x \leqslant b$

$$Q(x) = \int_c^d f(x, y)\,dy\dagger.$$

Substituting this value for $Q(x)$ in formula (3), we obtain

$$V = \int_a^b dx \int_c^d f(x, y)\,dy.$$

† This function of x is continuous [Sec. 296], which we presupposed in deriving formula (3).

But we also have the expression (2) for the volume V, hence,

$$\iint\limits_{(P)} f(x, y)\,dP = \int\limits_a^b dx \int\limits_c^d f(x, y)\,dy, \tag{4}$$

so the double integral has been reduced to a repeated one.

An analogous result can also be obtained for the more general case when the domain (P) on the plane xy is a curvilinear trapezoid, bounded by the two curves:

$$y = y_0(x), \quad y = Y(x) \quad (a \leqslant x \leqslant b)$$

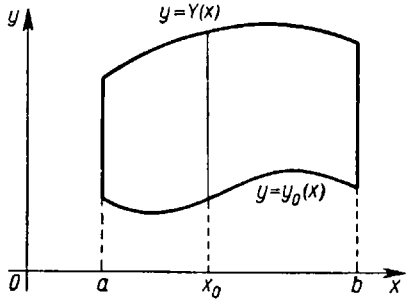

FIG. 21.

and the two ordinates $x = a$ and $x = b$ (Fig. 21). The difference, by comparison with the case considered, lies in the following: before, for any fixed $x = x_0$, the variation in y takes place in the one interval $[c, d]$, but now this interval

$$[y_0(x_0),\ Y(x_0)]$$

itself depends on x_0, so that

$$Q(x_0) = \int\limits_{y_0(x_0)}^{Y(x_0)} f(x_0, y)\,dy.$$

Finally we obtain

$$V = \iint\limits_{(P)} f(x, y)\,dP = \int\limits_a^b dx \int\limits_{y_0(x)}^{Y(x)} f(x, y)\,dy^{\dagger}. \tag{5}$$

† Here again the inner integral is a continuous function of x [see Sec. 299].

We have introduced the reader to the concept of a double integral and its calculation by a geometrical treatment.

338. The definition of a double integral. Now we come to a more general exposition of the problem—that from a purely analytical point of view. However, we will not avoid geometry here or, at least, geometrical language [Secs. 124–127]. We will speak of a "two-dimensional domain" (P); where a definite function of two variables is under consideration, the "curves" dividing it into partial "domains", we will take the "areas" of these "domains", etc. In fact these are the "domains" and "curves" from arithmetical two-dimensional space, and their "points" are pairs of numbers. But usually all these "images" are replaced for convenience by the actual geometrical images corresponding to them, and we make no distinction between them. In particular, by the "area of a domain" in arithmetical two-dimensional space we will always mean the area of the corresponding geometrical domain.

We recall that for the quadrature of a domain bounded by some curve, it is necessary and sufficient that this curve have an area equal to zero [Sec. 193]. A large class of such curves consists of continuous curves, expressed in terms of explicit equations, made up of a finite number of similar pieces. We note here (without embarking on the proof) that the same property, in particular, is also possessed by smooth or piece-wise smooth curves. *We will presuppose from now on that both the contour of the domain* (P), *and the curves, by which we break it up into pieces, all have zero area* (for instance, they belong to the class mentioned); this guarantees the existence of all the areas we need.

Now we shall return to the concept of a double integral, already introduced in Sec. 336, and we will give a more general definition of it in an expanded form.

In the domain (P) let there be defined a function $f(x, y)$†. We break up the domain (P) by a network of curves into a finite number of domains (P_1), (P_2), ..., (P_n), whose areas will be $P_1, P_2, \ldots, P_n$. Although for the greatest simplicity these partial domains are connected, yet to simplify later exposition it is advantageous not to

† Here we make no hypotheses as to its continuity.

exclude the possibility that they may be unconnected. Within the limits of the ith elementary domain (P_i) let us take an arbitrary point (ξ_i, η_i), and let the value of the function at this point be $f(\xi_i, \eta_i)$; we multiply this by the area P_i of the corresponding domain and add together all similar products. The sum obtained

$$\sigma = \sum_{i=1}^{n} f(\xi_i, \eta_i) P_i$$

we will call the *integral sum* for the function $f(x, y)$ in the domain (P).

We denote by λ the greatest of the diameters† of the partial domains (P_i). *The finite limit I of the integral sum σ as $\lambda \to 0$*

$$I = \lim \sigma ‡$$

is called the double integral *of the function $f(x, y)$ in the domain (P)* and is denoted by the symbol

$$I = \iint_{(P)} f(x, y) dP.$$

A function which has an integral is called *integrable.*

339. A condition for the existence of a double integral. An integrable function must necessarily be bounded. In fact, otherwise for any given way of breaking up the domain (P) into pieces, by the choice of the point (ξ_i, η_i) one could make the integral sum arbitrarily large, so that a finite limit I could not exist.

Turning to the consideration of conditions for the given function $f(x, y)$ to be integrable, we will therefore presuppose in advance that it is bounded:

$$m \leqslant f(x, y) \leqslant M.$$

As in the case of a function of one variable, it is also convenient here to introduce the so-called *Darboux upper and lower sums*:

$$s = \sum_{i=1}^{n} m_i P_i, \qquad S = \sum_{i=1}^{n} M_i P_i,$$

† The diameter of a set of points is precisely the upper bound of the distance between two arbitrary points of the set. In the case of a plane closed domain, bounded by a continuous curve, the diameter is simply the longest chord.

‡ The reader can easily determine for himself the exact meaning of this new limit.

where m_i and M_i denote, respectively, the exact lower and upper bounds of the values of the function $f(x, y)$ in the domain (P_i).

For a given way of breaking up the domain (P) into pieces, independently of the choice of the points (ξ_i, η_i), the inequality

$$s \leqslant \sigma \leqslant S$$

will be satisfied. But by a proper choice of these points the values $f(\xi_i, \eta_i)$ can be made as near as desired to $m_i(M_i)$, and together with this the sum σ can be made as near as desired to $s(S)$. Thus, *the Darboux upper and lower sums are, respectively, exactly the upper and lower bounds of the integral sums*, corresponding to the same decomposition of the domain.

For the Darboux sum, as in the linear case, we can establish the following properties.

FIRST PROPERTY. *If the domain (P_i) is broken up further by the addition of new lines of division to the previous ones, the lower Darboux sum does not decrease, and the upper one does not increase.*

SECOND PROPERTY. *Each lower Darboux sum does not exceed each upper sum, even though it may correspond to another way of breaking up the domain (P).*

The proof is analogous to the previous one [Sec. 177]; only in the cases when we spoke of points of division there, we now speak of lines of division.

There is, however, one point which we would like the reader to notice. In the linear case each new point of division clearly breaks up one of the old intervals into two; the common part of two intervals is again an interval. In the plane case the situation is complicated by the fact that two curves can intersect each other at many points and even at an infinite number of points. Therefore a connected partial domain can be broken up by a new curve into disconnected parts; in exactly the same way the common part of two connected domains may be a disconnected domain. This is why at the very beginning we did not exclude from our consideration the possibility of splitting up the fundamental domain into disconnected domains.

Further, the existence of the exact bounds

$$I_* = \sup\{s\}, \quad I^* = \inf\{S\},$$

is established, where it is shown that

$$s \leqslant I_* \leqslant I^* \leqslant S.$$

Finally, by means of a literal reproduction of the proof for the linear case [Sec. 178] we also obtain here

THEOREM. *For the existence of a double integral it is necessary and sufficient that*

$$\lim_{\lambda \to 0} (S - s) = 0,$$

or in the other notation

$$\lim_{\lambda \to 0} \sum_{i=1}^{n} \omega_i P_i = 0, \tag{6}$$

where ω_i is the oscillation $M_i - m_i$ of the function $f(x, y)$ in the partial domain (P_i).

340. Classes of integrable functions. With the aid of the test for integrability established above it is easy to prove:

I. *Every function $f(x, y)$, continuous in the domain (P), is integrable.*

In fact, if the function f is continuous in the (closed) domain (P), then by the property of uniform convergence, to every $\varepsilon > 0$ there corresponds a $\delta > 0$ such that in any part of the domain (P) with diameter less than δ the oscillation of the function will be less than ε. Now let the domain (P) be broken up into the parts (P_i), whose diameters are all less than δ. Then all the oscillations $\omega_i < \varepsilon$, and

$$\sum_i \omega_i P_i < \varepsilon \sum_i P_i = \varepsilon P,$$

whence it follows that condition (6) is also satisfied. This proves the integrability of the function.

Remark. Now it is easy to make the derivation of formula (2) for the volume of a cylindrical body completely rigorous. This is done in exactly the same way as for the derivation of the integral formula for the area of a curvilinear trapezoid [Sec. 196]—by using the bodies contained in and containing it, whose volumes are expressed by Darboux sums.

In order to extend somewhat the class of functions for which integrability has been established, we will need the following lemma.

LEMMA. *Let there be given in the domain* (P) *some curve* (L), *having area equal to zero. Then to every* $\varepsilon > 0$ *there corresponds a* $\delta > 0$ *such that, provided only that the domain* (P) *is broken up into pieces with diameters less than* δ, *the sum of the areas of those of them which have points in common with* (L) *will be less than* ε.

By the hypothesis the curve (L) can be embedded in a polygonal domain (Q) with area less than ε. We can do this in such a way that the curve (L) and the contour (K) of the given domain have no points in common. Then the distance between the variable points of both curves attains as its least value $\delta > 0$.

In fact, let the continuous curves indicated be given, respectively, by the parametric equations:

$$(K)\quad x = \varphi(t),\quad y = \psi(t);\qquad t_0 \leqslant t \leqslant T \qquad\qquad (L)\quad x = \varphi^*(u),\quad y = \psi^*(u),\qquad u_0 \leqslant u \leqslant U$$

where $\varphi, \psi, \varphi^*, \psi^*$ are continuous functions, each with respect to its argument. Then the distance between two arbitrary points of these curves

$$\sqrt{\{[\Phi(t) - \Phi^*(u)]^2 + [\psi(t) - \psi^*(u)]^2\}}$$

will be a continuous function of (t, u) in the closed rectangle $[t_0, T; u_0, U]$ and, hence, attains there its least value [Sec. 136]. In so far as the curves do not intersect, this least distance δ will be distinct from zero.

Now we break up the domain (P) arbitrarily into parts so that their diameters are less than δ. Those of them which contain the curve (L) will necessarily lie entirely within the domain (Q), hence their common area will be less than ε.

II. *If a bounded function* $f(x, y)$ *has, perhaps, discontinuities only on a finite number of curves with zero area, then it is integrable.*

Suppose we are given an arbitrary number $\varepsilon > 0$. By hypothesis every "linear discontinuity" of the function $f(x, y)$ can be enclosed inside a polygonal domain (Q) with common area less than ε. In Fig. 22 this domain is shaded. Its boundary is formed by the finite number of breaks (L), which, clearly, themselves have area equal to zero.

In the closed domain, obtained from (P) by excluding the interior of the domain (Q), the function $f(x, y)$ is continuous throughout,

and this means that it is also uniformly continuous. Hence, for a given $\varepsilon > 0$ we can find a number $\delta_1 > 0$ such that in every part of this domain, whose diameter is less than δ_1, the oscillation of the function $f(x, y)$ will be less than ε.

Now, by virtue of the lemma, we can also find a $\delta_2 > 0$ such that each time, as the domain (P) is broken up by arbitrary curves into parts with diameter less than δ_2, the sum of the areas of those of them which contain the whole set of breaks (L)—the boundary of the excluded polygonal domain (Q)—will certainly be less than ε.

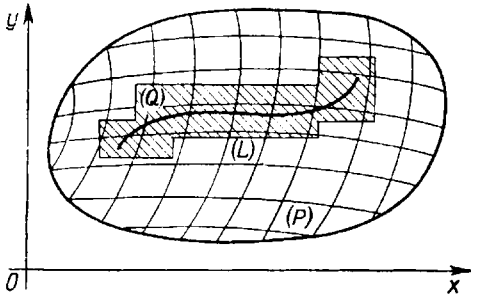

FIG. 22.

Let δ be the smaller of the two numbers δ_1, δ_2. We break up the domain (P) into the pieces $(P_1), (P_2), \ldots, (P_n)$, whose diameters are less than δ, and we consider the corresponding sum

$$\sum_i \omega_i P_i.$$

We split it into two sums:

$$\sum_{i'} \omega_{i'} P_{i'} + \sum_{i''} \omega_{i''} P_{i''},$$

presupposing that the symbol i' corresponds to those domains $(P_{i'})$ which lie entirely outside the excluded domain (Q), and the symbol i'' to all the others. We will calculate each of these sums separately.

Since all the $(P_{i'})$ lie in the domain obtained from (P) by excluding (Q), and since their diameters are less than $\delta \leqslant \delta_1$, we have that all the $\omega_{i'} < \varepsilon$, so that

$$\sum_{i'} \omega_{i'} P_{i'} < \varepsilon \sum_{i'} P_{i'} < \varepsilon P.$$

On the other hand, if we denote by Ω the oscillation of the function $f(x, y)$ in the whole domain (P), then we will have (because $\omega_i \leqslant \Omega$)

$$\sum_{i''} \omega_{i''} P_{i''} \leqslant \Omega \sum_{i''} P_{i''}.$$

Here $\sum P_{i''}$ is the sum of the areas of those of the domains (P_i), which (*1*) either lie entirely in the excluded domain (Q), or (*2*) contain the boundary (L) of this domain. The common area of the first is less than ε, because $Q < \varepsilon$; then the same can be said of the common area of the second, in so far as the domain is broken up into pieces with diameters less than $\delta \leqslant \delta_2$. Thus, $\sum P_{i''} < 2\varepsilon$, so that

$$\sum_{i''} \omega_{i''} P_{i''} < 2\Omega\varepsilon.$$

Finally, when $\lambda < \delta$, we obtain:

$$\sum_{i} \omega_i P_i < (P + 2\Omega)\varepsilon.$$

Since the first part of this inequality is arbitrarily small with ε, the condition (6) is satisfied, etc.

341. The properties of integrable functions and double integrals. *(1) If in an arbitrary way we vary the value of the function $f(x, y)$, integrable in (P), along any curve (L) with zero area (under the single condition that the variable function remains bounded), then again the function obtained is also integrable in (P), and its integral is equal to the integral of $f(x, y)$.*

For the proof we need to form the integral sum for the variable and the original function. They can differ only by those summands which belong to the domains (P_i), containing the curve (L). But, by the lemma in Sec. 340 the common area of these domains tends to zero as $\lambda \to 0$, from which it is easy to deduce that both integral sums tend to a common limit.

Thus, *the existence and value of a double integral does not depend on the values assumed by the functions under the integral along a finite number of curves with zero area.*

(2) If the domain (P) in which the function $f(x, y)$ is given is divided into two domains (P') and (P'') by the curve (L) with zero

area, then from the integrability of the function $f(x, y)$ in the whole domain (P) follows its integrability in the partial domains (P') and (P''), and conversely—from the integrability of the function in both the domains (P') and (P'') follows its integrability in the domain (P). Then here

$$\iint\limits_{(P)} f(x, y)\, dP = \iint\limits_{(P')} f(x, y)\, dP' + \iint\limits_{(P'')} f(x, y)\, dP''.$$

We split up the domains (P') and (P'') arbitrarily into parts; in this way (P) is also split up into parts:

$$(P_1),\ (P_2),\ \ldots,\ (P_n).$$

If the symbol i' denotes the parts lying in (P') and the symbol i'' the parts lying in (P'') then

$$\sum \omega_i P_i = \sum \omega_{i'} P_{i'} + \sum \omega_{i''} P_{i''}.$$

Let the function $f(x, y)$ be integrable in (P), so that as $\lambda \to 0$ the sum on the left tends to zero; then each of the sums on the right also certainly tends to zero, so that our function is also integrable in (P') and (P'').

Conversely, if the latter case holds, so that as $\lambda \to 0$ both the sums on the right tend to zero, then the sum on the left also tends to zero. However, we must understand that it is constructed not for an arbitrary division of the domain into parts, since we started from the separate division of the domains (P') and (P'').

In order to arrive at a division of the partial type from an arbitrary division of the domain (P), it is sufficient to add to the lines of division the curve (L). The sums corresponding to them will differ only by summands corresponding to those elementary domains which contain the curve (L). But, by the lemma in Sec. 340, their common area tends to zero as $\lambda \to 0$, and the two sums differ by an infinitely small amount. Thus, condition (6) is satisfied with complete generality, and the function $f(x, y)$ is shown to be integrable in (P).

Finally, the formula to be proved is obtained by taking limits as $\lambda \to 0$ from the equation

$$\sum f(\xi_i,\ \eta_i) P_i = \sum f(\xi_{i'},\ \eta_{i'}) P_{i'} + \sum f(\xi_{i''},\ \eta_{i''}) P_{i''}.$$

Analogously, from the consideration of the integral sums, by taking limits we obtain the following three properties:

(3) If we multiply a function $f(x, y)$, integrable in (P), by a constant k, then the function obtained will also be integrable, and here

$$\iint\limits_{(P)} kf(x, y)\, dP = k \iint\limits_{(P)} f(x, y)\, dP.$$

(4) If in the domain (P) the functions $f(x, y)$ and $g(x, y)$ are integrable, then the function $f(x, y) \pm g(x, y)$ is also integrable, where

$$\iint\limits_{(P)} [f(x, y) \pm g(x, y)]\, dP = \iint\limits_{(P)} f(x, y)\, dP \pm \iint\limits_{(P)} g(x, y)\, dP.$$

(5) If for the functions $f(x, y)$ and $g(x, y)$, integrable in (P), the inequality $f(x, y) \leqslant g(x, y)$ is satisfied, then

$$\iint\limits_{(P)} f(x, y)\, dP \leqslant \iint\limits_{(P)} g(x, y)\, dP.$$

Further,

(6) In the case when the function $f(x, y)$ is integrable the function $|f(x, y)|$ is also integrable, and the inequality

$$\left| \iint\limits_{(P)} f(x, y)\, dP \right| \leqslant \iint\limits_{(P)} |f(x, y)|\, dP$$

holds.

The integrability of the function $|f|$ follows from the simple remark that the oscillation ω_i^* of this function in any domain (P_i) does not exceed the corresponding oscillation ω_i of the function f. Thus

$$\sum \omega_i^* P_i \leqslant \sum \omega_i P_i,$$

and the fact that the second sum tends to zero implies that the first also tends to zero.

The required inequality is obtained by taking limits in the inequality

$$\left| \sum f(\xi_i, \eta_i) P_i \right| \leqslant \sum |f(\xi_i, \eta_i)|\, P_i.$$

(7) If the function $f(x, y)$, integrable in (P), satisfies the inequality

$$m \leqslant f(x, y) \leqslant M,$$

then

$$mP \leqslant \iint\limits_{(P)} f(x, y)\, dP \leqslant MP \tag{7}$$

This is obtained from the obvious inequality

$$mP \leqslant \sum f(\xi_i, \eta_i) P_i \leqslant MP$$

by taking limits.

If we divide all the parts of the inequality (7) by P:

$$m \leqslant \frac{\iint\limits_{(P)} f(x, y)\, dP}{P} \leqslant M$$

and denote by μ the mean ratio, then we obtain another form of the inequality (7)

$$\iint\limits_{(P)} f(x, y)\, dP = \mu P \qquad (m \leqslant \mu \leqslant M), \tag{8}$$

which expresses the so-called mean value theorem.

Now we presuppose, in particular, that the function $f(x, y)$ is continuous in (P), and we take as m and M its least and greatest values in the domain (P)—their existence follows from Weierstrass's theorem [Sec. 136]. Then by the well-known Bolzano–Cauchy theorem [Sec. 134] the continuous function $f(x, y)$, taking the values m and M, must pass through every intermediate value. Thus, in every case in the domain (P) we must be able to find a point $(\overline{x}, \overline{y})$ such that $\mu = f(\overline{x}, \overline{y})$, and the formula (8) takes the form:

$$\iint\limits_{(P)} f(x, y)\, dP = f(\overline{x}, \overline{y}) P. \tag{9}$$

This is a particularly common form of the mean value theorem.

In the same way it is easy to carry over the generalized mean value theorem to the case under consideration (Sec. 182, *(10)*]; we leave this to the reader.

342. An integral as an additive function of the domain; differentiation in the domain. We consider the (closed) plane domain *(P)* and the partial (closed) domains (p) contained in it. We will presuppose all the domains quadrable (under the circumstances they

may also be subject to other restrictions). *If every part* (p) *of the domain* (P) *is associated with some definite number*

$$\Phi = \Phi((p)),$$

then this defines a "function of the domain (p)*" for the given* (p). As examples of such a function of the domain we can cite: the area of a domain, a mass continuously distributed over it, the moments of this mass, a load continuously distributed over it or in general a force acting on it, etc.

If for an arbitrary division of the domain (p) *into non-overlapping parts*

$$(p) = (p') + (p'')$$

it can always be shown that

$$\Phi((p)) = \Phi((p')) = \Phi((p''))$$

then the function $\Phi((p))$ *of the domain is called additive.* All the functions introduced above in the form of examples possess this property of additiveness. Additive functions of a domain have a particular importance, because they are often encountered in the study of various aspects of its nature.

Let there be given in the quadrable domain (P) an integrable function of a point $f(M) = f(x, y)$; then it will be integrable in any quadrable part (p) of the domain, so that the integral

$$\Phi((p)) = \iint\limits_{(p)} f(x, y)\, dP \tag{10}$$

is also a function of the domain (p). In view of Sec. 341, *(2)* it will also, clearly, be an additive function.

Now we turn to "the differentiation of the function $\Phi((p))$ with the respect to domain". Let M be a fixed point of the domain (P), and (p) any partial domain containing this point. *The finite limit of the ratio*

$$\frac{\Phi((p))}{p},$$

where p *is the area of the domain* (p), *as the diameter of the domain* (p) *tends to zero, is called the* derivative of the function $\Phi((p))$ with respect to the domain *at the point* M. If $\Phi((p))$, for instance,

is a mass, continuously distributed over the plane figure (p), then $f(M)$ is none other than the density of the distribution of masses at the point M; if $\Phi((p))$ denotes a force, acting on the figure (p), then $f(M)$ expresses the specific pressure at the point M, etc.

We are particularly interested in the case when the function of the domain can be expressed by an integral of the form (10), where $f(x, y)$ is a continuous function in the domain (P). We will prove that *the derivative with respect to the domain at the point M of the integral will be the function under the integral, evaluated at this point, i.e.*

$$f(M) = f(x, y).$$

In fact, taking the domain (p), which is mentioned in the definition of the derivative, we have by the mean value theorem [see (9)]

$$\Phi((p)) = f(\bar{x}, \bar{y})p,$$

where $(\bar{x}, \bar{y})$ is some point of the domain (p). If the diameter of the domain (p) tends to zero, then the point $(\bar{x}, \bar{y})$ approaches (x, y) boundlessly and, by the continuity,

$$\frac{\Phi((p))}{p} = f(\bar{x}, \bar{y}) \to f(x, y),$$

which was what we wished to prove.

Remark. We found it convenient above to speak of additive functions of an interval [Sec. 204]. Because such a function is always the difference of two values of some function of a point, then there was no need in the "linear" case to develop the theory as in the study of the "plane" case above. However, in the theorem on the differentiation of a definite integral with respect to a variable upper limit [Sec. 183, *(12)*] the reader can easily see an analogy with what has been precisely proved in the theorem on the differentiation of a double integral with respect to the domain.

§2. The evaluation of a double integral

343. The reduction of a double integral to a repeated integral in the case of a rectangular domain. In Sec. 337 we have already discussed this problem by a geometrical treatment and under certain special hypotheses.

Now we will consider it by analytical methods and also in its general form; we begin with a simple case, viz. when the domain of integration is a rectangle $(P) = [a, b; c, d]$.

THEOREM. *If for the function $f(x, y)$, defined in the rectangle $(P) = [a, b; c, d]$, there exists the double integral*

$$\iint_{(P)} f(x, y)\,dP \tag{1}$$

and—for every constant value of x from $[a, b]$—the simple integral

$$I(x) = \int_c^d f(x, y)\,dy \quad (a \leqslant x \leqslant b), \tag{2}$$

there also exists the repeated integral

$$\int_a^b dx \int_c^d f(x, y)\,dy \tag{3}$$

and the equation

$$\iint_{(P)} f(x, y)\,dP = \int_a^b dx \int_c^d f(x, y)\,dy^{\dagger} \tag{4}$$

is satisfied.

Proof. We divide the intervals $[a, b]$ and $[b, c]$, defining the rectangle (P), into parts, taking the points of division

$$x_0 = a < x_1 < \ldots < x_i < x_{i+1} < \ldots < x_n = b,$$

$$y_0 = c < y_1 < \ldots < y_k < y_{k+1} < \ldots < y_m = d.$$

Then the rectangle (P) is divided into partial rectangles (Fig. 23):

$$(P_{i,k}) = [x_i, x_{i+1};\ y_k, y_{k+1}]$$

$$(i = 0, 1, \ldots, n-1;\ k = 0, 1, \ldots, m-1).$$

We denote by $m_{i,k}$ and $M_{i,k}$, respectively, the exact lower and upper bounds of the function $f(x, y)$ in the rectangle $(P_{i,k})$, so that for all points (x, y) of this rectangle

$$m_{i,k} \leqslant f(x, y) \leqslant M_{i,k}.$$

† The reader can easily see in this assertion a modification of the well-known theorem on double and repeated limits [Sec. 131].

Fixing x arbitrarily in the interval $[x_i, x_{i+1}]$: $x = \xi_i$, and integrating with respect to y from y_k to y_{k+1}, we will have [Sec. 182, (8)] where $\Delta y_k = y_{k+1} - y_k$; the integral with respect to y exists, because we have presupposed the existence of the integral (2) in the whole interval $[c, d]$. Summing similar inequalities with respect to k from 0 to $m-1$, we obtain

$$\sum_{k=0}^{m-1} m_{i,k}\,\Delta y_k \leqslant I(\xi_i) = \int_c^d f(\xi_i, y)\,dy \leqslant \sum_{k=0}^{m-1} M_{i,k}\,\Delta y_k.$$

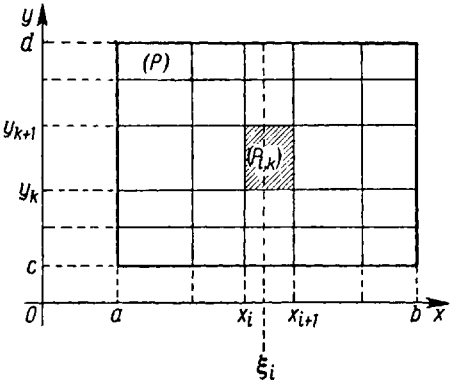

FIG. 23.

If we multiply all the parts of these inequalities by $\Delta x_i = x_{i+1} - x_i$ and sum with respect to the symbol i from 0 to $n-1$, then we find

$$\sum_{i=0}^{n-1} \Delta x_i \sum_{k=0}^{m-1} m_{i,k}\,\Delta y_k \leqslant \sum_{i=0}^{n-1} I(\xi_i)\,\Delta x_i \leqslant \sum_{i=0}^{n-1} \Delta x_i \sum_{k=0}^{m-1} M_{i,k}\,\Delta y_k.$$

In the middle term we have obtained the integral sum for the function $I(x)$. As for the extreme terms, they are just the Darboux sums s and S for the double integral (1). In fact, because $\Delta x_i \Delta y_k$ is the area $P_{i,k}$ of the rectangle $(P_{i,k})$, then, for instance, we have

$$\sum_{i=0}^{n-1} \Delta x_i \sum_{k=0}^{m-1} m_{i,k}\,\Delta y_k = \sum_{i=0}^{n-1} \sum_{k=0}^{m-1} m_{i,k}\,\Delta x_i\,\Delta y_k = \sum_{i,k} m_{i,k} P_{i,k} = s.$$

Thus, finally

$$s \leqslant \sum_{i=0}^{n-1} I(\xi_i)\Delta x_i \leqslant S.$$

If now all the Δx_i and Δy_k tend simultaneously to zero, then, in view of the existence of the double integral (1), both the sums s and S will tend to it as their limit. In that case also

$$\lim \sum_{i=0}^{n-1} I(\xi_i)\Delta x_i = \int\int_{(P)} f(x, y)\,dP,$$

i.e. the double integral (1) is at the same time also the integral of the function $I(x)$:

$$\int\int_{(P)} f(x, y)\,dP = \int_a^b I(x)\,dx = \int_a^b dx \int_c^d f(x, y)\,dy,$$

which is what we wished to prove.

By changing the roles of x and y, together with (4) we can also prove the formula

$$\int\int_{(P)} f(x, y)\,dP = \int_c^d dy \int_a^b f(x, y)\,dx; \tag{4*}$$

under the hypothesis that $y = \text{const}$ the integral

$$\int_a^b f(x, y)\,dx$$

exists.

Remark. If together with the double integral (1) there exist both the simple integrals:

$$\int_c^d f(x, y)\,dy (x = \text{const}) \quad \text{and} \quad \int_a^b f(x, y)\,dx (y = \text{const})$$

then both the formulae (4) and (4*) hold simultaneously, whence

$$\int_a^b dx \int_c^d f(x, y)\,dy = \int_c^d dy \int_a^b f(x, y)\,dx. \tag{5}$$

The application of formula (4) or (4*) is subject to the existence of the double integral and one of the simple ones. If the function

$f(x, y)$ is continuous (the case which is usually encountered in practice), then the existence of all the integrals mentioned is guaranteed. In this case any of the formulae mentioned can be used for the actual evaluation of the double integral, because the evaluation of the simple integrals is a much simpler problem.

In the proof of formula (4) the most natural thing of all was to divide the rectangle (P) by straight lines, parallel to the axes, into rectangular elements with areas $\Delta x_i \Delta y_k$. To indicate in the symbol of the double integral itself its origin from the division of the domain into parts by straight lines, parallel to the axes, instead of

$$\iint\limits_{(P)} f(x, y)\, dP$$

we often write

$$\iint\limits_{(P)} f(x, y)\, dx\, dy \qquad \left[\text{or} \quad \iint\limits_{(P)} f(x, y)\, dy\, dx\right].$$

Moreover, because we have in view the reduction of the double integral, extended over the rectangle $(P) = [a, b; c, d]$, to a repeated integral, the double integral itself is often denoted by a symbol like a repeated integral:

$$\int\limits_a^b \int\limits_c^d f(x, y)\, dy\, dx \qquad \text{or} \qquad \int\limits_c^d \int\limits_a^b f(x, y)\, dx\, dy.$$

In this notation the "outside integral" and the "inside integral" correspond to one another, so that it is only necessary to introduce the brackets in order to obtain one or the other of the repeated integrals:

$$\int\limits_a^b \left\{ \int\limits_c^d f(x, y)\, dy \right\} dx \qquad \text{or} \qquad \int\limits_c^d \left\{ \int\limits_a^b f(x, y)\, dx \right\} dy.$$

Examples. *(1)* $I = \int\limits_0^1 \int\limits_0^1 \frac{y\, dx\, dy}{(1 + x^2 + y^2)^{3/2}}$.

It is simpler to represent I by means of formula (4) in the form

$$I = \int\limits_0^1 dx \int\limits_0^1 \frac{y\, dy}{(1 + x^2 + y^2)^{3/2}}$$

because we at once obtain:

$$\int_0^1 \frac{y\,dy}{(1+x^2+y^2)^{3/2}} = \frac{1}{\sqrt{(x^2+1)}} - \frac{1}{\sqrt{(x^2+2)}},$$

so that

$$I = \int_0^1 \left(\frac{1}{\sqrt{(x^2+1)}} - \frac{1}{\sqrt{(x^2+2)}}\right) dx = \log \frac{x+\sqrt{(x^2+1)}}{x+\sqrt{(x^2+2)}}\Bigg|_0^1 = \log \frac{2+\sqrt{2}}{1+\sqrt{3}}.$$

If we have recourse to the other repeated integral, then the squaring process turns out to be somewhat more complicated:

$$I = \int_0^1 y\,dy \int_0^1 \frac{dx}{(1+x^2+y^2)^{3/2}},$$

$$\int_0^1 \frac{dx}{(1+x^2+y^2)^{3/2}} = \frac{1}{1+y^2}\,\frac{x}{\sqrt{(1+x^2+y^2)}}\Bigg|_{x=0}^{x=1} = \frac{1}{(1+y^2)\sqrt{(2+y^2)}},$$

$$I = \int_0^1 \frac{y\,dy}{(1+y^2)\sqrt{(2+y^2)}} = \frac{1}{2}\log \frac{\sqrt{(2+y^2)}-1}{\sqrt{(2+y^2)}+1}\Bigg|_0^1 = \frac{1}{2}\log \frac{(\sqrt{3}-1)(\sqrt{2}+1)}{(\sqrt{3}+1)(\sqrt{2}-1)}.$$

It is easy to transform this answer to the previous form.

(2) To find the volume V of the body, bounded below by the plane xy, with sides given by the planes $x = 0$, $x = a$, $y = 0$, $y = b$, and bounded above by the elliptical paraboloid

$$z = \frac{x^2}{2p} + \frac{y^2}{2q}.$$

First of all by formula (2)

$$V = \iint_{[0,\,a;\,0,\,b]} \left(\frac{x^2}{2p} + \frac{y^2}{2q}\right) dP.$$

The actual evaluation of the integral is performed by means of formula (4*):

$$V = \int_0^b dy \int_0^a \left(\frac{x^2}{2p} + \frac{y^2}{2q}\right) dx = \int_0^b \left(\frac{a^3}{6p} + \frac{ay^2}{2q}\right) dy = \frac{ab}{6}\left(\frac{a^2}{p} + \frac{b^2}{q}\right).$$

344. The reduction of a double integral to a repeated integral in the case of a curvilinear domain. We consider the domain (P), bounded below and above by two continuous curves:

$$y = y_0(x), \quad y = Y(x) \quad (a \leqslant x \leqslant b),$$

and with sides given by the two ordinates $x = a$ and $x = b$ (Fig. 24). Then, by analogy with the theorem in Sec. 343, there holds the following

THEOREM. *If for the function $f(x, y)$, defined in the domain (P), there exists the double integral*

$$\iint_{(P)} f(x, y)\,dP$$

and—for every constant value of x from $[a, b]$—the simple integral

$$I(x) = \int_{y_0(x)}^{Y(x)} f(x, y)\,dy,$$

then there also exists the repeated integral

$$\int_a^b dx \int_{y_0(x)}^{Y(x)} f(x, y)\,dy,$$

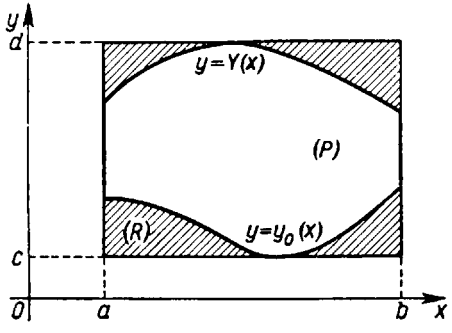

FIG. 24.

and the equation

$$\iint_{(P)} f(x, y)\,dP = \int_a^b dx \int_{y_0(x)}^{Y(x)} f(x, y)\,dy \tag{6}$$

is satisfied.

The proof is based on the reduction of this case to that considered in Sec. 343. In fact, we enclose the domain (P) in the rectangle

$$(R) = [a, b;\ c, d],$$

letting $c = \min\limits_{a \leqslant x \leqslant b} y_0(x)$, $d = \max\limits_{a \leqslant x \leqslant b} Y(x)$ (Fig. 24), and we define in this rectangle a function $f^*(x, y)$ in the following way:

$$f^*(x, y) = \begin{cases} f(x, y), & \text{if the point } (x, y) \text{ belongs to the domain } (P), \\ 0 & \text{at the other points of the rectangle } (R). \end{cases}$$

We will prove that this function satisfies the conditions of the theorem in Sec. 343.

First of all, it is integrable in the domain (P), because here it coincides with the function $f(x, y)$, which is integrable by hypothesis; clearly, therefore,

$$\iint\limits_{(P)} f^*(x, y)\, dP = \iint\limits_{(P)} f(x, y)\, dP.$$

On the other hand, $f^*(x, y) = 0$ outside (P) and, hence, is also integrable in the remaining part $(Q) = (R) - (P)$ of the rectangle (R)†, where

$$\iint\limits_{(Q)} f^*(x, y)\, dQ = 0.$$

Then, by virtue of Sec. 341, *(2)*, the function f^* is integrable in the whole rectangle (R), and

$$\iint\limits_{(R)} f^*(x, y)\, dR = \iint\limits_{(P)} f(x, y)\, dP. \tag{7}$$

For a constant value of x in $[a, b]$ there exists the integral

$$\int_c^d f^*(x, y)\, dy = \int_c^{y_0(x)} f^*\, dy + \int_{y_0(x)}^{Y(x)} f^*\, dy + \int_{Y(x)}^d f^*\, dy,$$

because each of the three integrals on the right exists. In fact, because in the intervals $(c, y_0(x))$ and $(Y(x), d)$ of variation of y the function $f^*(x, y) = 0$, the first and third integrals exist, and are equal to zero. The second integral coincides with the integral of the function $f(x, y)$:

$$\int_{y_0(x)}^{Y(x)} f^*(x, y)\, dy = \int_{y_0(x)}^{Y(x)} f(x, y)\, dy,$$

in so far as $f^*(x, y) = f(x, y)$ for y in $[y_0(x), Y(x)]$. Finally,

$$\int_c^d f^*(x, y)\, dy = \int_{y_0(x)}^{Y(x)} f(x, y)\, dy. \tag{8}$$

† Its values on the boundary of this domain play no part (see Sec. 341, *(1)*).

By virtue of the theorem mentioned, for the function f^* there also exists the repeated integral, which is equal to the double integral [see Sec. 343, (4)]:

$$\iint\limits_{(R)} f^*(x, y)\,dR = \int\limits_a^b dx \int\limits_c^d f^*(x, y)\,dy.$$

Taking (7) and (8) into account, we see that this formula is equivalent to formula (6).

If the domain (P) is a curvilinear trapezoid of the second kind and is bounded by the curves

$$x = x_0(y), \qquad x = X(y) \qquad (c \leqslant y \leqslant d)$$

and the straight lines $y = c$, $y = d$, then instead of (6) we arrive at the formula

$$\iint\limits_{(P)} f(x, y)\,dP = \int\limits_c^d dy \int\limits_{x_0(y)}^{X(y)} f(x, y)\,dx, \tag{6*}$$

under the hypotheses that the double integral and, also, the simple integral with respect to x when $y = \text{const}$ exist.

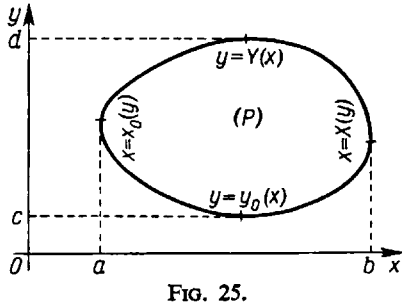

FIG. 25.

Remark. If the contour of the domain (P) is intersected only in two points both by the parallels to the ordinate axis and by the parallels to the abscissa axis (as, for instance, in the case illustrated in Fig. 25), then, when the given conditions are satisfied, both the formulae mentioned are applicable. By comparing them we obtain the equation

$$\int\limits_a^b dx \int\limits_{y_0(x)}^{Y(x)} f(x, y)\,dy = \int\limits_c^d dy \int\limits_{x_0(y)}^{X(y)} f(x, y)\,dx, \tag{9}$$

which has an independent interest. This is an analogue of formula (5) in Sec. 343.

If the function $f(x, y)$ is continuous in the domain (P), then the integrals, double and single, exist, and formula (6) or (6*), according to the type of domain (P), can be used to evaluate the double integral.

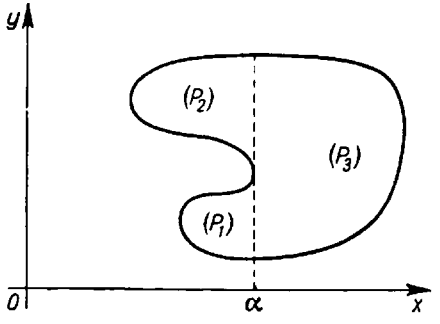

FIG. 26.

In the case of a more complicated contour the domain (P) is usually divided into a finite number of parts of the type considered. For instance, the figure (P) in Fig. 26 is divided by the straight line $x = \alpha$ into three such parts: (P_1), (P_2) and (P_3). Then the original integral, by virtue of Sec. 341, *(2)*, is the sum of the integrals extended separately over these parts; each of them can be evaluated as indicated.

In the general case also, in so far as we reduced the situation to the theorem in Sec. 343, the basis of the conclusion is the division of the figure under consideration into rectangular elements. In connection with this, here also we often use for the notation for the double integral the symbol

$$\iint\limits_{(P)} f(x, y)\, dx\, dy;$$

the product $dx\, dy$ represents the area of the elementary rectangle.

The notations

$$\int_a^b \int_{y_0(x)}^{Y(x)} f\, dy\, dx \quad \text{and} \quad \int_c^d \int_{x_0(y)}^{X(y)} f\, dx\, dy$$

are also perfectly clear.

Examples. (1) To evaluate the double integral

$$I = \iint\limits_{(P)} y^2 \sqrt{(R^2 - x^2)}\, dP,$$

where (P) is a circle of radius R with centre at the origin of the coordinates (Fig. 27).

Solution. The contour of the domain (P) has the equation $x^2 + y^2 = R^2$, whence $y = \pm\sqrt{(R^2 - x^2)}$. Clearly $y = +\sqrt{(R^2 - x^2)}$ is the equation of the upper semicircle, and $y = -\sqrt{(R^2 - x^2)}$ is the equation of the lower semicircle. Thus, for

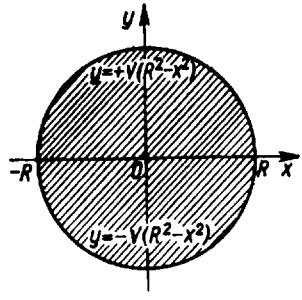

FIG. 27.

a constant x from the interval $[-R, R]$ the variable y changes from $-\sqrt{(R^2 - x^2)}$ to $+\sqrt{(R^2 - x^2)}$. By formula (6)—taking into account the parity with respect to y of the function under the integral—

$$I = \int\limits_{-R}^{R} dx \int\limits_{-\sqrt{(R^2-x^2)}}^{+\sqrt{(R^2-x^2)}} y^2 \sqrt{(R^2 - x^2)}\, dy = 2 \int\limits_{-R}^{R} \sqrt{(R^2 - x^2)}\, dx \int\limits_{0}^{\sqrt{(R^2-x^2)}} y^2\, dy.$$

We evaluate the inner integral:

$$\int\limits_{0}^{\sqrt{(R^2-x^2)}} y^2\, dy = \tfrac{1}{3}(R^2 - x^2)^{3/2}.$$

Then, again taking into account the parity,

$$I = \tfrac{2}{3} \int\limits_{-R}^{R} (R^2 - x^2)^2\, dx = \tfrac{4}{3} \int\limits_{0}^{R} (R^2 - x^2)^2 dx = \tfrac{32}{45} R^5.$$

The evaluation by means of formula (6*) is performed in exactly the same way.

(2) To evaluate the integral

$$I = \iint \sqrt{(4x^2 - y^2)}\, dx\, dy,$$

extended over the triangle which is formed by the straight lines $y = 0$, $x = 1$, $y = x$.

Solution. By formula (6)

$$I = \int_0^1 dx \int_0^x \sqrt{(4x^2 - y^2)}\, dy;$$

the inner integral is equal to

$$\int_0^x \sqrt{(4x^2 - y^2)}\, dy = \frac{y}{2}\sqrt{(4x^2 - y^2)} + 2x^2 \arcsin \frac{y}{2x}\Bigg|_{y=0}^{y=x} = \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right) x^2,$$

and finally

$$I = \frac{1}{3}\left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right).$$

The evaluation could also be carried out by means of formula (6*), but in this case we would encounter more difficult squares. This fact has always to be taken into account in the choice of the path for the evaluation.

(3) To evaluate the integral

$$I = \iint\limits_{\substack{x \geqslant 0,\ y \geqslant 0 \\ x + y \leqslant 1}} x^{p-1} y^{q-1}\, dx\, dy,$$

with the hypothesis that $p \geqslant 1$, $q \geqslant 1$†.

By formula (5) we have

$$I = \int_0^1 x^{p-1}\, dx \int_0^{1-x} y^{q-1}\, dy = \frac{1}{q}\int_0^1 x^{p-1}(1-x)^q\, dx = \frac{1}{q} B(p, q+1).$$

Finally:

$$\iint\limits_{\substack{x \geqslant 0,\ y \geqslant 0 \\ x + y \leqslant 1}} x^{p-1} y^{q-1}\, dx\, dy = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q+1)}.$$

This formula is due to Dirichlet.

(4) When $p < 1$ (or $q < 1$) the function under the integral becomes infinite when $x = 0$ and $0 < y \leqslant 1$ (or, respectively, when $y = 0$ and $0 < x \leqslant 1$). The usual definition of an integral is not applicable here: the integral turns out to be "improper" and it requires for its definition an additional passage to a limit.

† We only impose this restriction here in order to avoid the situation when the function under the integral becomes infinite; below, in *(4)*,' this restriction will be weakened.

We will clarify this in the example under consideration, with the hypothesis that $0<p<1$ and $0<q<1$. First let us take the integral ($\varepsilon>0$):

$$\iint\limits_{\substack{x\geqslant\varepsilon,\ y\geqslant\varepsilon\\ x+y\leqslant 1}} x^{p-1}y^{q-1}\,dx\,dy = \int\limits_{\varepsilon}^{1-\varepsilon} x^{p-1}\,dx \int\limits_{\varepsilon}^{1-x} y^{q-1}\,dy$$

$$= \frac{1}{q}\left[\int\limits_{\varepsilon}^{1-\varepsilon} x^{p-1}(1-x)^q\,dx - \varepsilon^q \int\limits_{\varepsilon}^{1-\varepsilon} x^{p-1}\,dx\right].$$

If we now make ε tend to 0 (by compressing the strip shaded in Fig. 28, with whose help we isolated the "singular" lines) then as the limit we obtain again $(1/q)B(p, q+1)$. This limit is the improper integral we are looking for.

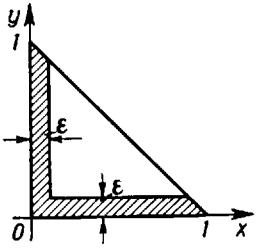

FIG. 28.

Remark. In the general case also, when the function under the integral becomes infinite at separate ("singular") points or along several ("singular") lines, we exclude them with the help of some neighbourhoods or other, we evaluate the integral in the domain without these neighbourhoods, and then we take limits, by compressing the given neighbourhoods to points or to lines. If a finite limit is shown to exist, then it also gives the value of the improper integral.

In a similar way we form the concept of an "improper" integral extended over an infinite domain: first we consider the proper integral over a bounded domain, and then this domain is extended, gradually encompassing all the points of the unbounded domain; the further passage to a limit is connected with this.

In the case of a positive function under the integral the choice of the neighbourhoods of the "singular" points and lines to be compressed or the bounded domain to be extended is irrelevant.

345. A mechanical application. All geometrical and mechanical magnitudes connected with a plane continuously distributed mass over some figure (P) and, representing additive functions of the domain, can be expressed in principle by double integrals extended over this figure.

Here we shall give a brief indication of how one usually obtains formulae of this type. The order of ideas here is the same as that in the application of a simple definite integral [Sec. 204].

Singling out an elementary part (dP) of the figure (P), we give the simplest estimate of its value, assuming, for instance, that the mass of the whole element is concentrated at one point or that the density of the distribution of the masses within the limits of the element is constant, which enables one to give an approximate expression of the form

$$dQ = q(M)dP$$

for the element dQ of the magnitude Q for which we are looking; it is true to within an infinitely small amount, greater than dP. Then the precise value of Q is expressed by the formula

$$Q = \iint\limits_{(P)} q(M)dP.$$

This can be justified (for instance cf. Sec. 204). By summing the approximate expressions for the elements dQ, we can obtain an approximate value for the magnitude Q in the form of an integral sum, and by taking limits we can obtain a precise value of Q in the form of a limit of a sum, i.e. an integral.

Let the masses be continuously distributed over the domain (P), where the (surface) density at the point $M(x, y)$ will be

$$\varrho(M) = \varrho(x, y).$$

It is easy to see that the element of mass will then be

$$dm = \varrho\, dP,$$

so that the whole mass is expressed by the integral

$$m = \iint\limits_{(P)} \varrho\, dP. \tag{10}$$

Further, the elementary statical moments and the moments of inertia with respect to the axes of coordinates will be

$$dK_x = y\, dm = y\varrho\, dP, \qquad dK_y = x\, dm = x\varrho\, dP,$$
$$dI_x = y^2\, dm = y^2\varrho\, dP, \qquad dI_y = x^2\, dm = x^2\varrho\, dP.$$

Hence for the moments themselves we obtain at once:

$$\left.\begin{aligned} K_x &= \iint\limits_{(P)} y\varrho\, dP, \qquad K_y = \iint\limits_{(P)} x\varrho\, dP, \\ I_x &= \iint\limits_{(P)} y^2\varrho\, dP, \qquad I_y = \iint\limits_{(P)} x^2\varrho\, dP. \end{aligned}\right\} \tag{11}$$

Now in the usual way we obtain the coordinates of the centre of gravity of the figure:

$$\xi = \frac{\iint\limits_{(P)} x\varrho\, dP}{m}, \qquad \eta = \frac{\iint\limits_{(P)} y\varrho\, dP}{m}. \tag{12}$$

In the case of a homogeneous figure, $\varrho = \text{const}$, these formulae can be simplified:

$$\xi = \frac{\iint\limits_{(P)} x\,dP}{P}, \quad \eta = \frac{\iint\limits_{(P)} y\,dP}{P}. \tag{13}$$

In individual simple cases, with the help of double integrals we can cover such questions with respect to bodies—for instance, cylindrical bodies.

Let there be given such a body, bounded by the surface $z = z(x, y)$, its projection (P) on the xy plane and the projective cylinder, whose generators are parallel to the z-axis. If, for instance, it is required to define the statical moment K_{xy} of a homogeneous body (for simplicity we presuppose the volume density equal to unity), then we represent this body as consisting of a series of elementary columns with base dP and height z. The statical moment of a column with respect to the xy plane is equal to its mass or—which in the given case is the same thing—to the volume $z\,dP$, multiplied by the distance of its centre of gravity from this plane, i.e. by $z/2$. Thus, the elementary statical moment is

$$dK_{xy} = \frac{1}{2} z^2\,dP,$$

whence, by summing all the columns, we obtain

$$K_{xy} = \frac{1}{2} \iint\limits_{(P)} z^2\,dP. \tag{14}$$

Analogously we can also establish the formulae

$$K_{zx} = \iint\limits_{(P)} yz\,dP, \quad K_{yz} = \iint\limits_{(P)} xz\,dP. \tag{14a}$$

Hence it is easy to obtain an expression for the coordinates ξ, η, ζ of the centre of gravity of the body

$$\xi = \frac{K_{yz}}{V} = \frac{\iint\limits_{(P)} xz\,dP}{V}, \text{ etc.}$$

In exactly the same way the formulae for the moments of inertia of the body are derived: I_z with respect to the z-axis and I_{yz}, I_{zx} with respect to the coordinate planes yz, zx:

$$I_z = \iint\limits_{(P)} (x^2+y^2)z\,dP, \quad I_{zx} = \iint\limits_{(P)} y^2 z\,dP, \quad I_{yz} = \iint\limits_{(P)} x^2 z\,dP, \tag{15}$$

where it is clear that $I_z = I_{zx} + I_{yz}$.

If the volume density ϱ of the distribution of masses were on dependent z, the double integral would not be sufficient and it would be necessary to turn to the triple integral [see Sec. 379].

Examples. (1) To find the centre of gravity of the part of the homogeneous ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1$$

contained in the first octant (Fig. 29).

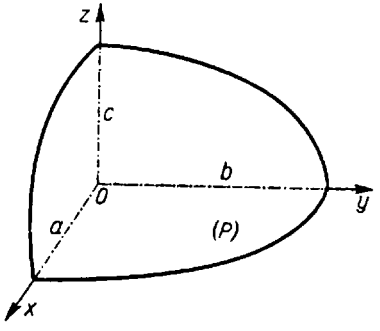

FIG. 29.

Solution. The domain (P) is bounded by the coordinate axes and the arc of the ellipse

$$y=\frac{b}{a}\sqrt{(a^2-x^2)}$$

$(0\leqslant x\leqslant a)$; the equation of the surface of the ellipsoid in explicit form will be

$$z=c\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}.$$

By formula (14)

$$K_{xy}=\frac{1}{2}c^2\int_0^a dx\int_0^{b\sqrt{(1-x^2/a^2)}}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dy$$

$$=\frac{bc^2}{3a^2}\int_0^a (a^2-x^2)^{3/2}\,dx=\frac{\pi}{16}abc^2.$$

Analogously

$$K_{yz}=\frac{\pi}{16}a^2bc,\qquad K_{zx}=\frac{\pi}{16}ab^2c.$$

At the same time the volume is

$$V=\frac{\pi}{6}abc,$$

so that

$$\xi=\frac{3}{8}a,\qquad \eta=\frac{3}{8}b,\qquad \zeta=\frac{3}{8}c.$$

(2) To find the moments of inertia of the homogeneous ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1$$

with respect to the coordinate planes.

Solution. We can limit ourselves to one octant of the ellipsoid (Fig. 29), provided we multiply the result by 8. In that case the domain (P) will be the quadrant of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1.$$

We have

$$I_{zx}=8\iint\limits_{(P)} y^2z\,dP=\frac{8c}{a}\int\limits_0^b y^2\,dy\int\limits_0^{a\sqrt{(1-y^2/b^2)}}\sqrt{\left[a^2\left(1-\frac{y^2}{b^2}\right)-x^2\right]}\,dx$$

$$=2\pi ac\int\limits_0^b y^2\left(1-\frac{y^2}{b^2}\right)dy=\frac{4}{15}\pi ab^3c.$$

Analogously

$$I_{yz}=\frac{4}{15}\pi a^3bc,\quad I_{xy}=\frac{4}{15}\pi abc^3.$$

§ 3. Green's formula

346. The derivation of Green's formula. In the present section we will establish an important formula, connecting double and curvilinear integrals.

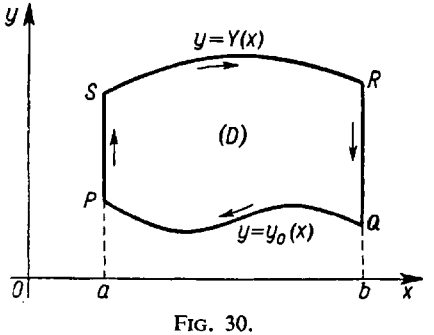

FIG. 30.

We consider the domain (D)—the "curvilinear trapezoid" (Fig. 30)—bounded by the contour (L), consisting of the curves

$$(PQ)\colon y=y_0(x)\,(a\leqslant x\leqslant b)\quad\text{and}\quad(SR)\colon y=Y(x)$$

and the two segments PS and QR of the parallels to the y-axis.

We presuppose that in the domain (D) there is given a function $P(x, y)$, continuous together with its derivative $\partial P/\partial y$.

Now we will evaluate the double integral

$$\iint\limits_{(D)} \frac{\partial P}{\partial y}\, dx\, dy$$

by means of formula (6) in Sec. 344; we obtain

$$\iint\limits_{(D)} \frac{\partial P}{\partial y}\, dx\, dy = \int\limits_a^b dx \int\limits_{y_0(x)}^{Y(x)} \frac{\partial P}{\partial y}\, dy.$$

The inner integral here is easy to evaluate with the help of the primitive function $P(x, y)$, namely:

$$\int\limits_{y_0(x)}^{Y(x)} \frac{\partial P}{\partial y}\, dy = P(x, y)\Big|_{y = y_0(x)}^{y = Y(y)} = P\big(x, Y(x)\big) - P\big(x, y_0(x)\big).$$

Thus,

$$\iint\limits_{(D)} \frac{\partial P}{\partial y}\, dx\, dy = \int\limits_a^b P\big(x, Y(x)\big)dx - \int\limits_a^b P\big(x, y_0(x)\big)dx;$$

each of these two integrals can now be replaced by a curvilinear integral. In fact, recalling formula (7), Sec. 331, we see that

$$\int\limits_a^b P\big(x, Y(x)\big)dx = \int\limits_{(SR)} P(x, y)dx,$$

$$\int\limits_a^b P\big(x, y_0(x)\big)dx = \int\limits_{(PQ)} P(x, y)dx.$$

Hence

$$\iint\limits_{(D)} \frac{\partial P}{\partial y}\, dx\, dy = \int\limits_{(SR)} P(x, y)dx - \int\limits_{(PQ)} P(x, y)\, dx$$

$$= \int\limits_{(SR)} P(x, y)dx + \int\limits_{(QP)} P(x, y)dx.$$

Because we wish to consider the integral along the whole contour (L) of the domain (D), we add to the right side of the equation obtained the integrals

$$\int\limits_{(PS)} P(x, y)\,dx \quad \text{and} \quad \int\limits_{(RQ)} P(x, y)\,dx;$$

clearly they are equal to zero, because the segments (PS) and (RQ) are perpendicular to the x-axis [see Sec. 331]. We obtain

$$\iint\limits_{(D)} \frac{\partial P}{\partial y}\,dx\,dy = \int\limits_{(PS)} P\,dx + \int\limits_{(SR)} P\,dx + \int\limits_{(RQ)} P\,dx + \int\limits_{(QP)} P\,dx.$$

The right side of this equation is the integral taken over the whole closed contour (L), bounding the domain (D), but in a negative direction. Corresponding to our agreement about the notation for curvilinear integrals along the closed contour [Sec. 332], we can finally write the formula obtained thus:

$$\iint\limits_{(D)} \frac{\partial P}{\partial y}\,dx\,dy = -\int\limits_{(L)} P(x, y)\,dx. \tag{1}$$

Although this formula has been derived under the hypothesis that the axes are right orientated, yet, as it easy to ascertain, it remains unchanged for a left orientation (provided only that we presuppose that the direction of going round the contour is reversed).

We have presupposed until now that the figure (D) is of the "curvilinear trapezoid" type, illustrated in Fig. 30, and the validity of formula (1) has been proved only for such domains. In fact, it is also valid for domains bounded by contours of a more complicated type, which may even consist of several separate curves. It is sufficient to presuppose that the figure (D) can be divided by straight lines, parallel to the y-axis, into a finite number of the "curvilinear trapezoids" we have mentioned [see, for instance, Fig. 26]. By writing down for each of them separately a formula of the type (1), we can add together all such equations. On the left we obtain the double integral extended over the whole domain (D), and on the right the sum of integrals taken over each of the partial contours. These latter, however, reduce to a single integral over the common contour (L), because the integrals over each of the auxiliary segments are equal to zero. Thus, in this case the formula (1) holds.

Analogously, we can also establish the formula

$$\iint_{(D)} \frac{\partial Q}{\partial x}\, dx\, dy = \int_{(L)} Q(x, y)\, dy \tag{2}$$

with the hypothesis that the function Q is continuous in the domain (D) together with its partial derivative $\partial Q/\partial x$. Here we take first, as the domain (D), a curvilinear trapezoid of the type illustrated in Fig. 31. It is bounded by the curves

$$\begin{aligned} &(PS): x = x_0(y) \\ &(QR): x = X(y) \end{aligned} \qquad (c \leqslant y \leqslant d)$$

and the two segments (PQ) and (SR), parallel to the x-axis. Then we can generalize the formula, as above, to the case of a domain which can be divided by straight lines, parallel to the x-axis, into a finite number of curvilinear trapezoids of this type.

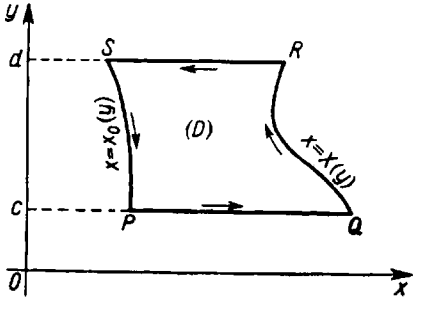

FIG. 31.

Finally, if the domain (D) simultaneously satisfies the conditions of both cases, i.e. it can be divided both into a finite number of trapezoids of the first type, and also (independently of this) into a finite number of trapezoids of the second type, then both the formulae (1) and (2) are valid for it, of course, with the hypothesis that the functions P, Q and their derivatives $\partial P/\partial y$, $\partial Q/\partial x$ are continuous. Subtracting formula (1) from (2), we obtain

$$\int_{(L)} P\, dx + Q\, dy = \iint_{(D)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy. \tag{3}$$

This is *Green's formula*†.

Remark. We can restate the conditions under which formula (3) is valid in a form which is easier to see: *Green's formula holds for any domain (D) bounded by one or several piece-wise smooth contours.* We shall not prove this.

347. An expression for area by means of curvilinear integrals. Below we will frequently come across applications of Green's formula; now we introduce one of the simplest of them—an expression for area.

If the functions P and Q in formula (3) are chosen so that the expression $\partial Q/\partial x - \partial P/\partial y$ becomes equal to one, then the double integral reduces to the area D of the figure (D), and we obtain an expression for this area with the help of some curvilinear integral, taken along the contour (L) of the figure. Thus, letting $Q = x$, $P = 0$, we find

$$D = \int_{(L)} x\,dy. \tag{4}$$

When $Q = 0$, $P = -y$, we obtain

$$D = -\int_{(L)} y\,dx. \tag{5}$$

The most useful formula, however, will be that corresponding to $Q = x/2$, $P = -y/2$:

$$D = \frac{1}{2}\int_{(L)} x\,dy - y\,dx. \tag{6}$$

Examples. (1) To find the area of the ellipse with semi-axes a and b.

We use the parametric equations of the ellipse: $x = a\cos t$, $y = b\sin t$ $(0 \leqslant t \leqslant 2\pi)$. By formula (6)

$$D = \frac{1}{2}\int_0^{2\pi} a\cos t \cdot b\cos t\,dt - b\sin t\,(-a\sin t)\,dt = \frac{ab}{2}\int_0^{2\pi} dt = \pi ab.$$

† George Green (1793–1841) was an English mathematician. In fact, however, formula (3) is not Green's, and it is connected with Green's name only in view of the analogy with other formulae deduced by Green for the three-dimensional case. Formula (3) was used by Gauss and Riemann, but in a special form it had already occurred in works on analysis in the eighteenth century.

To evaluate the curvilinear integral we applied formula (6), Sec. 331; in the ordering of the limits of integration we took into account that the positive circuit of the contour corresponds to the increase in the parameter.

(2) To find the area of the loop of the Descartes folium (Fig. 32)

$$x^3 + y^3 = 3axy.$$

To obtain the parametric equations of the contour we set $y = tx$†.

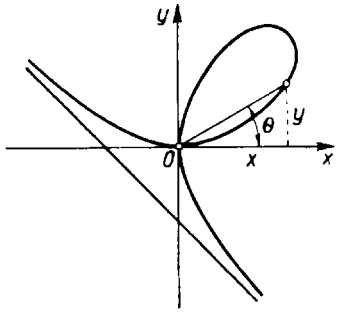

FIG. 32.

Then

$$x = \frac{3at}{1+t^3}, \quad y = \frac{3at^2}{1+t^3}.$$

From geometrical considerations it is clear that the loop is described as the parameter t varies from 0 to ∞ (because $t = y/x = \tan\theta$, where θ varies from 0 to $\pi/2$). We have

$$dx = 3a\frac{1-2t^3}{(1+t^3)^2}dt, \quad dy = 3a\frac{2t-t^4}{(1+t^3)^2}dt$$

and

$$D = \frac{9a^2}{2}\int_0^\infty \frac{t^2dt}{(1+t^3)^2} = \frac{3}{2}a^2.$$

We note that here we used an improper integral with an infinite limit, at the same time as in the derivation of formula (6), Sec. 331 we took into account that the interval of variation of the parameter is finite. It is easy to justify this action, if as a preliminary we introduce another parameter with finite interval of variation (for instance, the angle θ), and then go over to the parameter $t = y/x$.

† Such a substitution is convenient, as a rule, in such cases, when in the equation of an algebraic curve there are two homogeneous groups of terms where the degrees of these groups differ by one.

§ 4. Conditions for a curvilinear integral to be independent of the path of integration

348. The integral along a simple closed contour. Green's formula enables us to explore with ease the whole question of curvilinear integrals of the second type, which we only touched on casually in § 2 of Chapter 20 [see, for instance, the remark in Sec. 333].

Let there be given continuous functions $P(x, y)$ and $Q(x, y)$ in a connected domain (E). We will consider first *the problem of the vanishing of the integral*

$$\int_{(L)} P\,dx + Q\,dy$$

along any simple closed contour (L) *lying in* (E)†.

Having the application of Green's formula in mind, we will need the additional hypothesis that the continuous derivatives $\partial P/\partial y$ and $\partial Q/\partial x$ exist in (E). But this means that it is necessary that, *together with the contour* (L), *the domain* (D), *bounded from without by this contour, should always belong to the domain* (E). In other words, the domain (E) must not have any "holes", not even point-holes. A connected domain possessing this property is called *simply connected.*

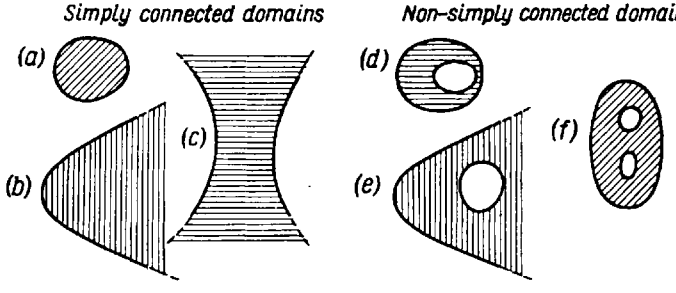

FIG. 33.

If we are discussing a finite domain, i.e. one which does not extend to infinity, then the concept of being simply connected can be formulated even more simply: *the domain must be bounded by a unique closed contour.* In Fig. 33 there are some examples of

† Every curve which is discussed in the present paragraph is presupposed to be piece-wise smooth.

simply connected and non-simply connected domains, of which (*a*), (*d*), (*f*) are finite, and (*b*), (*c*), (*e*) extend to infinity.

Now we will formulate our fundamental assertion:

THEOREM 1. *Let the functions $P(x, y)$ and $Q(x, y)$ be defined and continuous in the simply connected domain (E) together with their derivatives $\partial P/\partial y$ and $\partial Q/\partial x$. In order that the equation*

$$\int_{(L)} P\,dx + Q\,dy = 0 \tag{1}$$

should hold, for any simple closed contour (L) in the domain (E), it is necessary and sufficient that in (E)

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{A}$$

should hold identically.

In fact, by Green's formula, equation (1) is equivalent to the equation

$$\iint_{(D)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = 0. \tag{2}$$

The sufficiency of condition (A) for this to be fulfilled is clear. To prove the necessity, we presuppose that equation (2) is always valid. By differentiating the integral over the domain [Sec. 342], in view of the continuity of the function under the integral, we arrive at the conclusion that in (*E*)

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0, \quad \text{i.e.} \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

holds identically.

349. The integral along a curve joining two arbitrary points. We turn, finally, to *the question of the conditions which guarantee the independence of the integral*

$$\int_{(AB)} P\,dx + Q\,dy, \tag{3}$$

taken along the curve (AB), joining two points A and B of the domain (E), of the form of this curve. Here the decisive part will be played by condition (A).

THEOREM 2. *As a preliminary hypothesis, to ensure that the integral* (3) *should not depend on the path of integration, it is necessary and sufficient that condition* (A) *be satisfied*†.

Necessity. Let us suppose that the integral (3) does not depend on the path, and we introduce into (E) an arbitrary simple closed contour (L) (Fig. 34). If A and B are two of its points, then, by our assumption,

$$\int_{(AIB)} = \int_{(AIIB)}, \tag{4}$$

whence

$$\int_{(L)} = \int_{(AIB)} + \int_{(BIIA)} = 0. \tag{5}$$

Hence, by Theorem 1, (A) must necessarily be satisfied.

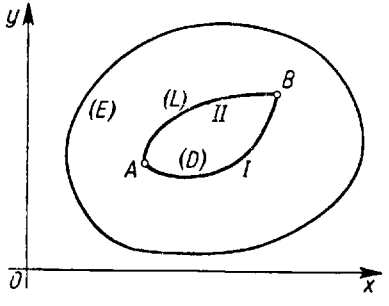

FIG. 34.

Sufficiency. Now we presuppose that equation (A) holds in (E). We need to prove that, if we join any points A, B of this domain by two simple curves (AIB) and $(AIIB)$, then equation (4) will always be valid. It is easy to do this if the given curves have no points in common, except A and B, because then the contour $(L) = (AIBIIA)$ will be simple, so that, by Theorem 1, (5) is satisfied, and from it, conversely, follows (4).

If the curves (AIB) and $(AIIB)$ intersect in a finite number of points, then the contour (L) will not be simple: it intersects itself

† The reader can easily verify that in examples *(1)* and *(4)*, Sec. 333 condition (A) is satisfied, and in examples *(2)*, *(3)* and *(5)* it is not.

in the points mentioned (Fig. 35). Starting from the point A and following the direction of the curve (L), we describe the part of it up to the first self-intersection—at the point C. Rejecting the closed curve (L_1) obtained, we continue the path up to a new self-intersection, which enables us to exclude yet another closed curve (L_2), and so on. After a finite number of steps the curve (L) turns out to factorize into a finite number of simple (not self-intersecting) closed curves

$$(L_1), (L_2), \ldots$$

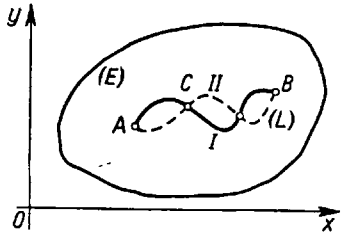

FIG. 35.

along which the integral is known to be zero. This means that it is also equal to zero along the whole curve (L), and hence (4) again follows.

However, the curves (AIB) and $(AIIB)$, in general, may intersect an infinite number of times, and in such a case the preceding reasoning does not apply. In order to overcome the difficulty encountered, we prove the following lemma on the approximation to a curvilinear integral by means of the integral taken along a broken curve.

LEMMA. Let the functions $P(x, y)$ and $Q(x, y)$ be continuous in some open domain (E), and (L) a non-closed piece-wise smooth curve, contained in (E). If we inscribe in (L) a broken curve (Λ), then as the greatest length of its links tends to zero we have

$$\lim \int_{(\Lambda)} P\,dx + Q\,dy = \int_{(L)} P\,dx + Q\,dy.$$

It is sufficient to limit ourselves to the integrals $\int_{(\Lambda)} P\,dx$ and $\int_{(L)} P\,dx$; for the integrals $\int_{(\Lambda)} Q\,dy$ and $\int_{(L)} Q\,dy$ the reasoning is completely analogous. Let the broken curve (Λ) inscribed in (L) have vertices at the points

$$A \equiv A_0, A_1, \ldots, A_i, A_{i+1}, \ldots, A_n \equiv B;$$

we denote by x_i, P_i the values of x, P at the point A_i. Given an arbitrary number $\varepsilon > 0$, it is possible to make the lengths of the chords $\overline{A_iA_{i+1}}$ small enough that: *(1)* the oscillation of the continuous function P along the link $\overline{A_iA_{i+1}}$ will be less than ε and *(2)* the integral sum $\sum_i P_i \Delta x_i$ differs from its limit $\int_{(L)} P dx$ also by less than ε.

We have, clearly,

$$\int_{(A)} P\,dx = \sum_i \int_{(A_iA_{i+1})} P\,dx$$

and, on the other hand,

$$\sum_i P_i \Delta x_i = \sum_i \int_{(A_iA_{i+1})} P_i\,dx,$$

so that

$$\int_{(A)} P\,dx = \sum_i P_i \Delta x_i + \sum_i \int_{(A_iA_{i+1})} [P - P_i]\,dx.$$

But the first summand on the right differs from the integral $\int_{(L)} P dx$ by less than ε [see *(2)*], and the second in its absolute value does not exceed $\varepsilon \sum_i \overline{A_iA_{i+1}}$ [see *(1)*], i.e. it is certainly less than $L\varepsilon$, where L is the length of the curve (L).

Thus, finally,

$$\left| \int_{(A)} P dx - \int_{(L)} P dx \right| < \varepsilon(1 + L),$$

which proves our assertion.

Now we shall return to our proof of the sufficiency. We inscribe in the curves (AIB) and $(AIIB)$, respectively, the broken curves (A_I) and (A_{II}); they can intersect in at most a finite number of points, and then—as we proved above—we will have

$$\int_{(AI)} = \int_{(AII)}.$$

It remains only to take limits in this equation; by taking into consideration that all the links in both the broken curves tend to zero, equation (4) is again established.

350. The connection with the problem of exact differentials. The differential expression

$$P dx + Q dy \tag{6}$$

represents the expression for the (complete) differential of the function $F(x, y)$ of two variables [Sec. 142]

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy,$$

which coincides with (2), if we set

$$\frac{\partial F}{\partial x} = P, \quad \frac{\partial F}{\partial y} = Q. \tag{7}$$

However, certainly not every expression of the form (2) is an "exact differential", i.e. not every such expression has a "primitive function" $F(x, y)$, for which in fact this expression serves as the (complete) differential. To solve the question of whether the expression (7) will be an exact differential or not, we again encounter the condition (A):

THEOREM 3. *With the previous hypotheses, in order that the expression* (6) *should be an exact differential, it is necessary and sufficient that condition* (A) *should be satisfied.*

The necessity is clear at once: if the expression (6) is the (complete) differential of some function $F(x, y)$, so that equation (7) holds, then

$$\frac{\partial P}{\partial y} = \frac{\partial^2 F}{\partial x\, \partial y}, \quad \frac{\partial Q}{\partial x} = \frac{\partial^2 F}{\partial y\, \partial x},$$

and it remains only to refer to the theorem in Sec. 147 on mixed derivatives (taking into account the hypothesis of their continuity).

We go on to the proof of the sufficiency. We already know that when condition (A) is satisfied the integral (3) turns out to be independent of the path of integration (by Theorem 2). In this case the integral is uniquely defined by the given points $A(x_0, y_0)$ and $B(x_1, y_1)$, in connection with which it can be denoted by the symbol

$$\int_A^B P\,dx + Q\,dy \quad \text{or} \quad \int_{(x_0, y_0)}^{(x_1, y_1)} P\,dx + Q\,dy.$$

Here only the beginning and end of the path of integration are indicated; the path itself is not indicated, but it makes no difference—we can integrate along any path.

If we fix the point $A(x_0, y_0)$, and we replace the point B by an arbitrary point $M(x, y)$ of the domain (E), then the integral obtained is some function of the point M, i.e. of its coordinates x, y, in the domain (E):

$$F(x, y) = \int_{(x_0, y_0)}^{(x, y)} P\,dx + Q\,dy. \tag{8}$$

We will show that this precise function will be a primitive for the expression (2), and for this purpose we will consider the question of its partial derivatives both with respect to x, and with respect to y.

Taking an arbitrary point $B(x_1, y_1)$ in the domain (E), we add to x_1 the increment Δx and go over to the point $C(x_1 + \Delta x, y_1)$, which for sufficiently small Δx will also belong to (E) together with the whole segment BC (Fig. 36). The corresponding values of the function will be

$$F(x_1, y_1) = \int_{(x_0, y_0)}^{(x_1, y_1)} P\,dx + Q\,dy,$$

$$F(x_1 + \Delta x, y_1) = \int_{(x_0, y_0)}^{(x_1 + \Delta x, y_1)} P\,dx + Q\,dy.$$

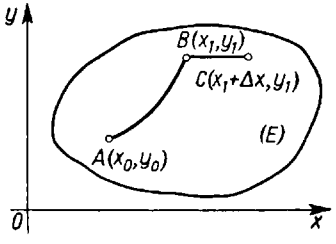

FIG. 36.

The first of these integrals we take along an arbitrary curve (K), joining the points A and B, and for the second integral we form the path of integration from this same curve and from the rectilinear segment BC. Thus, the increment in the function F will be

$$F(x_1 + \Delta x, y_1) - F(x_1, y_1) = \int_{(BC)} P\,dx + Q\,dy = \int_{(BC)} P(x, y)\,dx;$$

the integral containing $Q\,dy$ vanishes, because the segment BC is perpendicular to the y-axis.

The remaining integral reduces at once to an ordinary definite integral: for this we need to replace y by y_1 in the function under the integral (from the equation $y = y_1$ of the straight line BC) and

to take as the limits of integration with respect to x the abscissae of the points B and C. Finally

$$F(x_1+\Delta x, y_1)-F(x_1, y_1)=(\mathrm{R})\int_{x_1}^{x_1+\Delta x} P(x, y_1)\,dx.$$

Applying the mean value theorem to the ordinary integral obtained and dividing both sides of the equation by Δx, we find

$$\frac{F(x_1+\Delta x, y_1)-F(x_1, y_1)}{\Delta x}=P(x_1+\theta\Delta x, y_1) \quad (0\leqslant\theta\leqslant 1).$$

Now we make Δx tend to zero. By virtue of the continuity of the function $P(x, y)$ the right side of the equation, and with it also the left, tends to $P(x_1, y_1)$. Hence, at the point (x_1, y_1) the partial derivative of the function F with respect to x exists and is expressed by the equation

$$\frac{\partial F(x_1, y_1)}{\partial x}=P(x_1, y_1).$$

Analogously we can also establish the formula

$$\frac{\partial F(x_1, y_1)}{\partial y}=Q(x_1, y_1).$$

Because the point (x_1, y_1) was chosen arbitrarily inside the domain (E), then for all points of this domain the relation (7) will be satisfied

$$\frac{\partial F}{\partial x}=P, \quad \frac{\partial F}{\partial y}=Q.$$

In so far as these partial derivatives are continuous, *the function* $F(x, y)$ *has the differential* [Sec. 142]:

$$dF=\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy=P\,dx+Q\,dy,$$

which is what we wished to prove.

By comparing Theorems 1, 2 and 3 we can now derive the

COROLLARY. *With the given hypotheses, in order that the integral* (3) *should not depend on the path (and the integral along a closed contour should be equal to zero), it is necessary and sufficient that the expression* (6) *under the integral should be an exact differential.*

In the proof of Theorem 3, we have established for the curvilinear integral (8), which does not depend on its path, a result completely analogous to the theorem on the differentiation of an ordinary definite integral with respect to a variable upper limit [Sec. 183, *(12)*].

Now we presuppose that we know some primitive function $\Phi(x, y)$ for the expression (6) under the integral, so that—together with the relation (7), which holds for the function (8)—it will also be true that

$$\frac{\partial \Phi}{\partial x} = P, \quad \frac{\partial \Phi}{\partial y} = Q;$$

then the difference is

$$F(x, y) - \Phi(x, y) = C = \text{const}$$

(in fact its partial derivatives both with respect to x, and with respect to y, are identically equal to zero). Letting $x = x_0$, $y = y_0$, we find that $C = -\Phi(x_0, y_0)$, so that

$$F(x, y) = \Phi(x, y) - \Phi(x_0, y_0).$$

If, finally, we take here $x = x_1$, $y = y_1$, then we arrive at the formula

$$\int_{(AB)} P\,dx + Q\,dy = \Phi(x_1, y_1) - \Phi(x_0, y_0) = \Phi(x, y)\Big|_{x_0, y_0}^{x_1, y_1} \tag{9}$$

or, more briefly,

$$\int_{(AB)} P\,dx + Q\,dy = \Phi(B) - \Phi(A) = \Phi(M)\Big|_A^B. \tag{9a}$$

This formula is completely analogous to the fundamental formula of integral calculus [Sec. 185], expressing an ordinary definite integral in terms of its primitive. We note, however, once more, that it applies only to a curvilinear integral which does not depend on its path.

Remark. The characterization of the exact differential by the condition *(A)* is due to Euler and Claireau (1740).

The curvilinear integral (without this term) of the differential expression (6) is also first met in a work of Claireau. *The Theory of the Form of the Earth* ... (1743)†. Presupposing that the form of a curve is given by means of an equation between x and y, Claireau, with its help, excluded y and dy from (6), and then integrated the expression obtained, which contained only x and dx. At the same time Claireau indicated the condition for the independence of the integral of the form of the curve: *the expression* (6) *must be a complete differential,* and for this, in turn, *condition* (A) *must be satisfied.*

351. Applications to physical problems. We turn now to considering the application of the theory just presented to the problems examined earlier [Sec. 335] from the realm of mechanics and physics.

† There is a Russian translation (publ. Acad. Nauk SSSR, 1947); see p. 43.

(1) The work done by a field of force. We have seen that the work done by a field of force as a particle of mass 1 moves from the position A to the position B can be expressed by the curvilinear integral [see Sec. 335, (12)]:

$$A = \int_{(AB)} X\,dx + Y\,dy, \tag{10}$$

where $X = X(x, y)$ and $Y = Y(x, y)$ are the projections of the tension of the field on the coordinate axes, and (AB) denotes the trajectory of the particle.

It is very natural to wish to clarify the conditions under which the work of a field of force depends only on the first and last positions of the particle, but not on the form of the trajectory. This question, clearly, is equivalent to the question of the independence of the value of the curvilinear integral (10) of the path of the integration. Therefore the desired condition is the equation with the

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x} \tag{11}$$

hypothesis, of course, that the domain encompassing the field is simply connected and that for the functions X, Y and their derivatives continuity is nowhere violated.

The same condition can also be expressed in the following form: *the work done by a field of force as a particle moves from one position to another does not depend on the form of the trajectory if and only if the element of work*

$$X\,dx + Y\,dy$$

is a (complete) differential of some function $U(x, y)$. This function is usually called the *potential*; in the case when it exists the field itself is called *potential.*

The work of a potential field as a particle moves from the position $A(x_0, y_0)$ to the position $B(x_1, y_1)$ is equal [see Sec. 350, (9)] simply to the corresponding increment in the function of force:

$$U(x_1, y_1) - U(x_0, y_0) = U(B) - U(A). \tag{12}$$

As an example we consider the field of Newtonian attraction. If at the origin of coordinates O there is located a mass μ, and at the point A a mass 1, then this latter will be attracted to the centre O by a force $\vec{F}$, equal in magnitude to

$$F = \frac{\mu}{r^2},$$

where $r = \sqrt{(x^2 + y^2)}$ is the distance of the point A from the origin†. Because the cosines of the angles made by this force with the axes will be, respectively, $-x/r$ and $-y/r$, then the projection of the force $\vec{F}$ on the axes can be expressed thus:

$$X = -\frac{\mu x}{r^3}, \quad Y = -\frac{\mu y}{r^3}.$$

† It is the same for the *electrostatic field of Coulomb attraction,* created by a charge μ, located at the origin (if we consider the effect of the field on the charged particle).

It is immediately clear that the Newtonian field is a potential field, in so far as the expression

$$-\frac{\mu x}{r^3}\,dx-\frac{\mu y}{r^3}\,dy \tag{13}$$

is the differential for the function

$$U=\frac{\mu}{r},$$

which also plays here the part of the potential function; it is called the *Newtonian potential* (of the field of the point O). Notwithstanding the presence of a "singular" point at the origin of coordinates—there the functions X, Y suffer a discontinuity—the integral of the expression (13) along a closed contour will be zero, even if the contour surrounds the origin.

As the point moves from the position A to the position B the field of force does the work [see (12)]

$$A=\frac{\mu}{r_B}-\frac{\mu}{r_A},$$

where r_A and r_B are the distances of the points A and B from the centre.

(2) The flow of an incompressible fluid in a plane. If we denote by u, v the components along the axes of the vector velocity, then, as we saw in Sec. 335, *(2)*, the quantity of fluid flowing in unit time past the closed contour (K) on the inside will be equal to

$$Q=\int_{(K)} v\,dx-u\,dy$$

[see Sec. 335, (14)]. In the absence of sources and outlets this integral will always be zero. Hence it follows that the components u, v of the vector velocity are necessarily subject to the condition

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0.$$

Then the expression $v\,dx-u\,dy$ under the integral has the primitive function $\varphi(M)=\varphi(x, y)$, which in hydromechanics is called the *function of the current*.

If we take any curve (AB) joining the points A and B, then, as we know, [Sec. 335, (14)], the quantity of fluid flowing past it in unit time on a definite side can be expressed by the integral

$$Q=\int_{(AB)} v\,dx-u\,dy,$$

where the direction on the curve (AB) must be such that the normal directed towards the given side makes the angle $+\pi/2$ with the positive direction of the tangent. Now we see that this magnitude is equal simply to the difference $\varphi(B)-\varphi(A)$ of the values of the function of the current at the ends of the curve.

§ 5. Change of variables in double integrals

352. Transformation of plane domains. We presuppose that we are given two planes, one referred to the rectangular axes x and y, and the other to the rectangular axes ξ and η. We consider in these planes two bounded closed domains: the domain (D) in the xy plane and the domain (Δ) in the $\xi\eta$ plane.

The contour or boundary of each of these domains we will suppose to be a simple piece-wise smooth curve; we will denote it by the symbol (S) for the domain (D) and by the symbol (Σ) for the domain (Δ) (Fig. 37).

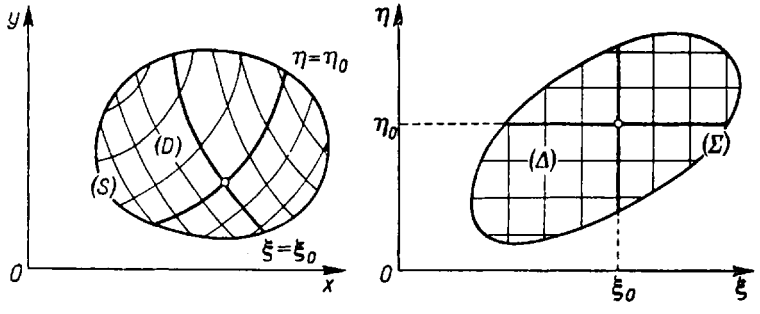

FIG. 37.

Let us suppose that in the domain (Δ) there is given a system of continuous functions:

$$\left.\begin{aligned} x &= x(\xi, \eta), \\ y &= y(\xi, \eta), \end{aligned}\right\} \tag{1}$$

which associate with each point (ξ, η) of the domain (Δ) one definite point (x, y) of the domain (D), where not one point (x, y) of (D) will be omitted, so that every such point is associated with at least one point (ξ, η) of (Δ).

If to distinct points (ξ, η) correspond distinct points (x, y) (which we will suppose from now on), so that each of the points (x, y) corresponds to only one point (ξ, η), then equations (1) *can be solved uniquely with respect to* ξ *and* η: the variables ξ, η in turn are single-valued functions of x, y in the domain (D)

$$\left.\begin{aligned} \xi &= \xi(x, y), \\ \eta &= \eta(x, y). \end{aligned}\right\} \tag{1a}$$

Thus we have established between the domains (D) and (Δ) a *mutually single-valued* or *one–one* correspondence. We also say that the formulae (1) realize *a transformation of the domain* (Δ) *into the domain* (D), and the formulae (1a) give *an inverse transformation of the domain* (D) *into the domain* (Δ).

We emphasize that here, necessarily, *to the points of the contour* (Σ) *correspond precisely the points of the contour* (S), and conversely.

We presuppose further that the functions (1) are not only continuous, but also have continuous partial derivatives of the first order in (Δ).

Then the functional determinant

$$\frac{D(x, y)}{D(\xi, \eta)} = \begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{vmatrix} \tag{2}$$

is also a continuous function of ξ, η in the domain (Δ). We will assume that *this determinant is always distinct from zero*, and hence, by the continuity, *is of constant sign.* This hypothesis will play an important part in what follows.

If we take a simple piece-wise smooth curve (Λ) in the domain (Δ), then by (1) it transforms into a similar curve (L) in the domain (D).

In fact, let the equations of the curve (Λ) be:

$$\xi = \xi(t), \quad \eta = \eta(t) \qquad (\alpha \leqslant t \leqslant \beta \quad \text{or} \quad \alpha \geqslant t \geqslant \beta), \tag{3}$$

where (if we limit ourselves to a smooth piece of the curve) we can assume that the functions $\xi(t)$, $\eta(t)$ have continuous derivatives, which do not vanish simultaneously. Substituting these functions in the transformation formulae (1), we obtain the parametric equations of the corresponding curve (L):

$$\begin{aligned} x &= x(\xi(t), \eta(t)) = x(t), \\ y &= y(\xi(t), \eta(t)) = y(t). \end{aligned} \tag{4}$$

It is easy to see that these functions also have continuous derivatives:

$$\left.\begin{aligned} x'(t) &= \frac{\partial x}{\partial \xi}\,\xi'(t) + \frac{\partial x}{\partial \eta}\,\eta'(t), \\ y'(t) &= \frac{\partial y}{\partial \xi}\,\xi'(t) + \frac{\partial y}{\partial \eta}\,\eta'(t), \end{aligned}\right\} \tag{5}$$

which also cannot vanish simultaneously, so that there are no singular points on the curve (L).

In fact, otherwise, by virtue of the fact that the determinant $D(x, y)/D(\xi, \eta)$ does not equal zero, it would follow from (5) that $\xi' = 0$ and $\eta' = 0$ simultaneously, which contradicts our hypothesis.

If the point (ξ, η) in the plane $\xi\eta$ traces out a closed contour (Δ), say, in the positive direction, then the corresponding point (x, y) also traces out some closed contour (L) in the plane xy, but *its direction may turn out to be either positive or negative.* The problem depends, as we shall see below [Sec. 354, *(1)*], on the sign of the functional determinant.

A given pair of values of the variables ξ and η from the domain (Δ) uniquely define some point in the domain (D) in the xy plane, and conversely. This is the reason for our calling the numbers ξ, η *the coordinates of the points of the domain* (D).

A curve, made up of points of the domain *(D)*, for which one of the coordinates has a constant value, is called a *coordinate line.*

For instance, letting $\eta = \eta_0$ in (1), we obtain a parametric representation of a coordinate line:

$$\begin{aligned} x &= x(\xi, \eta_0), \\ y &= y(\xi, \eta_0) \end{aligned}$$

(the part of the parameter is here played by ξ). We obtain an implicit equation of the same line by setting $\eta = \eta_0$ in the second of equations (2):

$$\eta(x, y) = \eta_0.$$

Since the coordinate lines, generally speaking, will be curves, the numbers ξ, η, which characterize the position of the point on the plane xy are called *the curvilinear coordinates of the point.*

Adding to the coordinate η distinct (possible) constant values, we obtain a whole family of coordinate lines in the xy plane. Fixing the value of the coordinate ξ, we obtain another family of coordinate lines. Because there exists a one–one correspondence between the domains under consideration, distinct lines of one family do not intersect amongst themselves, and through any point of the domain (D) passes one line of each family.

Each net of coordinate lines in the plane xy is the image of a net of straight lines $\xi = \text{const}$ and $\eta = \text{const}$ in the plane $\xi\eta$ (Fig. 37).

Examples. (1) A very simple and important example of curvilinear coordinates is given by *the polar coordinates* r, θ. They have an intuitive geometrical image, as the polar radius-vector and the polar angle, but they can also be introduced formally, with the help of the known relations:

$$\left.\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta. \end{aligned}\right\} \quad (r \geqslant 0)$$

If the values of r and θ are set out along two mutually perpendicular axes, taking, say, r to be the abscissa and θ the ordinate (with a right orientation of the axes), then to every point in the semi-plane $r \geqslant 0$ corresponds by the given formulae one definite point of the plane xy.

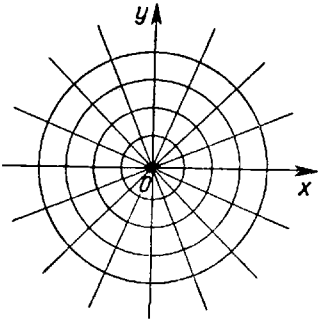

FIG. 38.

The reader will certainly have come across a discussion of the coordinate lines with respect to this case: to the straight lines $r = \text{const}$ correspond the circles with radius r and with centre at the origin, and to the straight lines $\theta = \text{const}$ correspond the rays extending from the origin and making an angle θ with the x-axis (Fig. 38).

However, in the given case the transformation formulae, in general, will not have a unique solution: a change of $2k\pi$ (where k is an integer) in the magnitude

of the angle θ does not affect the values of x and y. In order to obtain all the points of the xy plane, it is sufficient to limit ourselves to the values

$$r \geqslant 0, \quad 0 \leqslant \theta < 2\pi.$$

Every point (x, y), distinct from the origin, corresponds to one value of $r > 0$ and one value of θ in the given limits. But we have not succeeded in eliminating the violation of the single-valued correspondence at the origin of coordinates: to the point $x = y = 0$ corresponds the whole θ-axis in the $r\theta$ plane (or, if desired, its segment from $\theta = 0$ to $\theta = 2\pi$).

In the plane $r\theta$ we consider the closed rectangle $[0, R; 0, 2\pi]$ or $o\alpha\beta\gamma$ (Fig. 39); it is easy to see that, in the xy plane, there corresponds to it the closed circle described around the origin O with radius $R = OA$. But the whole contour of this circle corresponds to only one side $\alpha\beta$ of the given rectangle; to the sides $o\alpha$ and $\beta\gamma$ (both of them) corresponds precisely one radius OA of the circle; finally, to the whole side $o\gamma$ corresponds only the one point O. Here it is clear that the conditions given in the preceding paragraph are not satisfied.

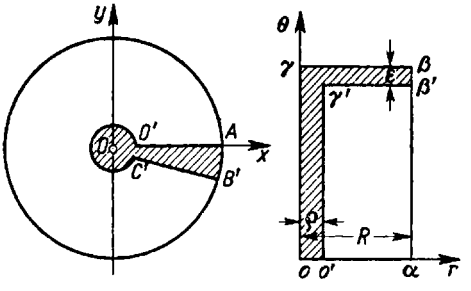

FIG. 39.

However, if we displace the side $o\gamma$ by a small quantity $\varrho = oo'$, and the side $\gamma\beta$ by $\varepsilon = \beta\beta'$, then the new rectangle $o'\alpha\beta'\gamma'$ will correspond to the figure $O'AB'C'$ in the xy plane, obtained from the circle by removing a small circle of radius ϱ and a sector with central angle ε, which satisfies all the requirements. As the point in the $r\theta$ plane varies along the segments $\alpha\beta'$, $\beta'\gamma'$, $\gamma'o'$, $o'\alpha$ the corresponding point in the xy plane describes in order: the incomplete circumference AB' (of radius R), the segment $B'C'$, the incomplete circumference $C'O'$ (of radius ϱ) and the segment $O'A$. We note in passing that the positive circuit in the xy plane corresponds to the positive circuit in the $r\theta$ plane.

The functional determinant in the given case is equal to

$$\frac{D(x, y)}{D(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r;$$

it keeps the positive sign (if we exclude the origin).

(2) Sometimes it is convenient to give a net of coordinate lines in advance and to establish a system of curvilinear coordinates with respect to them.

We consider, for instance, the two families of parabolas (Fig. 40):

$$y^2 = 2px \quad \text{and} \quad x^2 = 2qy;$$

each of them separately fills the whole xy plane (if we exclude the coordinate axes).

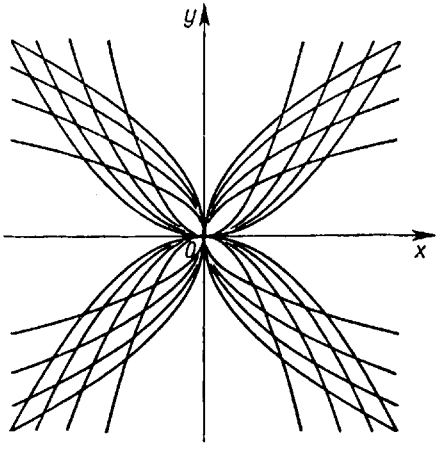

FIG. 40.

It is natural to introduce $\xi = 2p$ and $\eta = 2q$ as curvilinear coordinates. From the equations $y^2 = \xi x$ and $x^2 = \eta y$ we have

$$x = \sqrt[3]{}(\xi\eta^2), \quad y = \sqrt[3]{}(\xi^2\eta) \quad \text{and} \quad \xi = \frac{y^2}{x}, \quad \eta = \frac{x^2}{y} \qquad (x, y \neq 0).$$

The determinant here is equal to

$$\frac{D(x,y)}{D(\xi,\eta)} = \begin{vmatrix} \frac{1}{3}\xi^{-\frac{2}{3}}\eta^{\frac{2}{3}} & \frac{2}{3}\xi^{\frac{1}{3}}\eta^{-\frac{1}{3}} \\ \frac{2}{3}\xi^{-\frac{1}{3}}\eta^{\frac{1}{3}} & \frac{1}{3}\xi^{\frac{2}{3}}\eta^{-\frac{2}{3}} \end{vmatrix} = -\frac{1}{3}.$$

353. An expression for area in curvilinear coordinates. We presuppose that in the xy plane there is given some domain (D), bounded by a simple piece-wise smooth contour (S). Let the formulae (1) establish a one-one correspondence between this domain and the domain (Δ) in the $\xi\eta$ plane, bounded by a similar contour (Σ).

We retain all the hypotheses of Sec. 352 with respect to this transformation of the domains and, in addition, we also presuppose that

there exist in the domain (Δ) continuous mixed derivatives of the second order for any of the functions (1), say:

$$\frac{\partial^2 y}{\partial \xi \, \partial \eta} \quad \text{and} \quad \frac{\partial^2 y}{\partial \eta \, \partial \xi}.$$

[by virtue of the continuity, they will have equal values, Sec. 147]†.

With these hypotheses we set ourselves the problem of expressing the area D of the domain under consideration in the xy plane in the form of a double integral, extended over the domain Δ in the $\xi\eta$ plane.

We will start from the formula expressing the area D in terms of the curvilinear integral taken along the contour (S) of the domain (D)

$$D = \int_{(S)} x \, dy \tag{6}$$

[see Sec. 347, (4)].

The plan of the remaining transformations is as follows: first, by using the parametric equations of the contour, we will go over from the curvilinear integral (6) to an ordinary definite integral. Then we transform this latter again to a curvilinear integral, but taken this time along the contour (Σ) of the domain (Δ). Finally, using Green's formula, we change the curvilinear integral obtained to a double integral over the domain (Δ).

In the execution of this plan we need the parametric equations of the contour (S). Because in what follows we have in view the transformation to the contour (Σ), we shall now prefer to start from the equations of this contour. Let (3) give the parametric representation of the curve (Σ); then (4) clearly gives a similar representation for the curve (S), in so far as [which follows from our hypotheses, Sec. 352] it corresponds precisely in the xy plane to the curve (Σ). We choose the limits α and β of variation of t in such a way that in going from α to β the curve (S) is described in the positive direction; this can always be done.

Then, according to formula (5*), Sec. 331,

$$D = \int_{\alpha}^{\beta} x(t) y'(t) \, dt$$

† We note here that these additional hypotheses are not essential for the validity of the final result and are introduced only to simplify the proof.

or, if we take (4) and (5) into account,

$$D = \int_{\alpha}^{\beta} x(\xi(t), \eta(t)) \left[\frac{\partial y}{\partial \xi} \xi'(t) + \frac{\partial y}{\partial \eta} \eta'(t) \right] dt. \tag{7}$$

We compare this integral with the curvilinear integral

$$\int_{(\Sigma)} x(\xi, \eta) \left(\frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta \right), \tag{8}$$

taken along the contour (Σ) in the positive direction. If we wish to reduce the latter by the usual rule to an ordinary definite integral, then we would need to substitute here instead of ξ and η the functions $\xi(t)$ and $\eta(t)$ from the parametric equations of the curve (Σ), and we would return to the integral (7).

However, we need to bear one fact in mind. As t varies from α to β the contour (S) is described in a positive direction—for the way we have chosen the limits. But the contour (Σ) may here be described either in a positive or in a negative direction; thus, the integrals (7) and (8) may in fact differ in sign. In each case

$$D = \pm \int_{(\Sigma)} x \frac{\partial y}{\partial \xi} d\xi + x \frac{\partial y}{\partial \eta} d\eta, \tag{9}$$

where (we emphasize this yet again) *the plus sign occurs, if the positive circuit of the contour* (Σ) *corresponds to the positive circuit of the contour* (S), *and the minus sign occurs in the opposite case.*

It remains, finally, to transform the curvilinear integral obtained into a double integral. For this we need to use Green's formula

$$\int_{(\Sigma)} P(\xi, \eta) d\xi + Q(\xi, \eta) d\eta = \iint_{(\Delta)} \left(\frac{\partial Q}{\partial \xi} - \frac{\partial P}{\partial \eta} \right) d\xi d\eta,$$

where we set

$$P(\xi, \eta) = x \frac{\partial y}{\partial \xi}, \quad Q(\xi, \eta) = x \frac{\partial y}{\partial \eta}.$$

Because

$$\frac{\partial Q}{\partial \xi} = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} + x \frac{\partial^2 y}{\partial \eta \partial \xi},$$

$$\frac{\partial P}{\partial \eta} = \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi} + x \frac{\partial^2 y}{\partial \xi \partial \eta},$$

and the mixed derivatives of the second order of y are equal to each other, then

$$\frac{\partial Q}{\partial \xi} - \frac{\partial P}{\partial \eta} = \frac{D(x, y)}{D(\xi, \eta)},$$

and we arrive at the formula

$$D = \pm \int\int_{(\Delta)} \frac{D(x, y)}{D(\xi, \eta)} d\xi\, d\eta.$$

We saw in Sec. 352 that with the given hypotheses the determinant

$$J(\xi, \eta) = \frac{D(x, y)}{D(\xi, \eta)}$$

has a fixed sign in the domain (Δ). The integral has the same sign. But we have before us the double sign $\pm$; because as a result we must obtain the existence of a positive number D, then it is clear that *the sign before the integral coincides with the sign of the determinant.* If we introduce this sign into the function under the integral, then we clearly obtain there the absolute value of the determinant, so that the final expression for the area will be

$$D = \int\int_{(\Delta)} \left| \frac{D(x, y)}{D(\xi, \eta)} \right| d\xi\, d\eta = \int\int_{(\Delta)} |J(\xi, \eta)|\, d\xi\, d\eta. \tag{10}$$

This is the formula which we wished to establish.

The expression under the integral

$$\left| \frac{D(x, y)}{D(\xi, \eta)} \right| d\xi\, d\eta = |J(\xi, \eta)|\, d\xi\, d\eta$$

is usually called *the element of area in curvilinear coordinates.* We saw, for instance, that in the case of the transition to polar coordinates the functional determinant is equal to r, hence the element of area in polar coordinates is $r\, dr\, d\theta$.

354. Additional remarks. *(1)* If we compare the rule by which we selected the sign, plus or minus, in formula (9), with the fact that this sign necessarily coincides with the sign of the functional determinant, then we obtain the interesting property: *if this determinant has the positive sign, then the positive directions of the circuits*

of the contours (S) *and* (Σ) *correspond one to the other in the transformation formulae, but if the determinant has the negative sign, then the positive direction along one contour corresponds to the negative direction along the other.*

Clearly, this also holds with respect to any pair of mutually corresponding simple closed contours (L) and (Λ), lying in the domains (S) and (Σ). The result obtained can easily be verified for the examples cited in Sec. 352.

(2) Applying the mean value theorem [Sec. 341, (9)] to the formula (10), we obtain the relation

$$D = |J(\bar{\xi}, \bar{\eta})|\,\Delta, \tag{11}$$

where $(\bar{\xi}, \bar{\eta})$ is some point from the domain (Δ), and Δ is the area of this domain.

Comparing this relation with the Lagrange formula

$$f(\beta) - f(\alpha) = f'(\bar{\xi})(\beta - \alpha) \qquad (\alpha < \bar{\xi} < \beta).$$

If $x = f(\xi)$ is a monotonic function, it relates the interval $\alpha \leqslant \xi \leqslant \beta$ to the interval $f(\alpha) \leqslant x \leqslant f(\beta)$ in a one-one manner (or $f(\beta) \leqslant x \leqslant f(\alpha)$, if $f(x)$ is a decreasing function). We denote the lengths of these intervals by δ and d; then the Lagrange formula reduces to the equation

$$d = |f'(\bar{\xi})|\,\delta, \tag{12}$$

which resembles equation (11).

If we "compress" the interval $[\alpha, \beta]$ to the point ξ in the formula (12), then as a result we obtain the relation

$$|f'(\xi)| = \lim \frac{d}{\delta},$$

so that the absolute value of the derivative is, as it were, the *coefficient of expansion* of the straight line ξ (at the given point on it) in its transformation to the straight line x.

In exactly the same way from formula (11) by "compressing" the domain (Δ) to the point (ξ, η) we obtain

$$|J(\xi, \eta)| = \lim \frac{D}{\Delta}\dagger,$$

† In essence what we do is to differentiate the integral (10) over the domain at the point (ξ, η) [Sec. 342].

so that the absolute value of the functional determinant plays the part of the *coefficient of expansion* of the $\xi\eta$ plane (at the given point on it) in its transformation to the xy plane.

This remark emphasizes once again the strong analogy between the derivative and the functional determinant [cf. Chapter 19].

(3) Formula (10) shows that as the area Δ decreases without limit the area D corresponding to it also decreases without limit. Hence it is easy to establish that the transformation of the domains studied in Sec. 352 also possesses the following important property: *it maps a curve* (Λ) *with area equal to zero in the domain* (Δ) *into some curve* (L) *in the domain* (D), *which also has zero area.*

(4) Formula (10) was derived under the hypotheses that there is a one-one correspondence between the domains (D) and (Δ), that the functions (1) and their partial derivatives are continuous, and also that the determinant (2) keeps its sign unchanged. However, in practice we usually encounter cases where these hypotheses are violated at separate points or along separate curves.

If the given points and curves in both planes can be enclosed in domains (d) and (δ) with arbitrarily small areas, then when they are excluded the formula remains applicable:

$$D - d = \int\limits_{(\Delta)-(\delta)} \int |J(\xi, \eta)|\, d\xi\, d\eta. \tag{10a}$$

Let the functional determinant in the domain (Δ) remain bounded:

$$|J(\xi, \eta)| \leqslant M;$$

then the integral (10a) differs from the integral (10) by the quantity

$$\iint\limits_{(\delta)} |J(\xi, \eta)|\, d\xi\, d\eta \leqslant M\delta.$$

Taking limits in (10a) as d and $\delta \to 0$, we again establish formula (10).

By way of illustration we shall return to Example *(1)* of Sec. 352 and to the diagram shown in Fig. 39. It is not possible to apply formula (10) immediately to the rectangle $\Delta = [0, R; 0, 2\pi]$ and to the circle (D) of radius R (with centre at the origin); in this case the formula takes the form

$$D = \iint\limits_{(\Delta)} r\, dr\, d\theta,$$

But if we exclude the shaded domains (whose areas tend to zero together with ϱ and ε), then we can apply this formula to the domains obtained; it only remains to take limits.

355. A geometrical derivation. We derived formula (10) by means of arguments which, although they were simple, were not intuitive. We now mention another derivation of this formula, which is perfectly clear in its geometrical aspect; it is due to Ostrogradski [see below Sec. 359].

We again consider the transformation of the $\xi\eta$ plane into the xy plane, which is given by the formulae (1). We introduce in the $\xi\eta$ plane an infinitely small rectangle $\Pi_1\Pi_2\Pi_3\Pi_4$ with sides $d\xi$ and $d\eta$, parallel to the axes of ξ and η (Fig. 41*a*). The image of this rectangle in the xy plane is the curvilinear quadrangle $P_1P_2P_3P_4$ (Fig. 41*b*); we will determine its area.

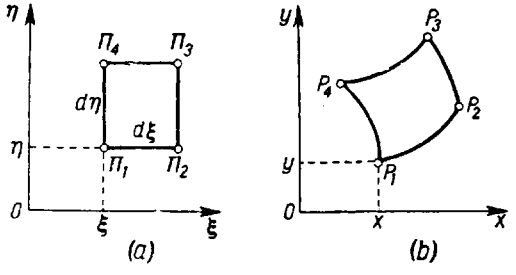

FIG. 41.

The vertices of the rectangle in the $\xi\eta$ plane have the coordinates

$$\Pi_1(\xi, \eta), \quad \Pi_2(\xi + d\xi, \eta), \quad \Pi_3(\xi + d\xi, \eta + d\eta), \quad \Pi_4(\xi, \eta + d\eta);$$

in this case the corresponding vertices of the curvilinear quadrangle will have the following coordinates:

$$P_1(x(\xi, \eta), y(\xi, \eta)),$$

$$P_2(x(\xi + d\xi, \eta), y(\xi + d\xi, \eta)),$$

$$P_3(x(\xi + d\xi, \eta + d\eta), y(\xi + d\xi, \eta + d\eta)),$$

$$P_4(x(\xi, \eta + d\eta), y(\xi, \eta + d\eta)).$$

If we limit ourselves to terms of the first order with respect to $d\xi, d\eta$, then we can take the points to be approximately:

$$P_1(x, y), \quad P_2\left(x + \frac{\partial x}{\partial \xi} d\xi, \quad y + \frac{\partial y}{\partial \xi} d\xi\right),$$

$$P_3\left(x + \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta, \quad y + \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta\right)$$

$$P_4\left(x + \frac{\partial x}{\partial \eta} d\eta, \quad y + \frac{\partial y}{\partial \eta} d\eta\right),$$

where $x = x(\xi, \eta), y = y(\xi, \eta)$ and, in general, all the derivatives are calculated at the point (ξ, η). Because the projections of the segments P_1P_2 and P_3P_4 on both axes are respectively equal, then these segments are equal and parallel, so that to within small quantities of higher order the quadrangle $P_1P_2P_3P_4$ is a parallelogram.

Its area is equal to double the area of the triangle $P_1P_2P_3$. From analytical geometry it is known that double the area of the triangle whose vertices are located at the points (x_1, y_1), (x_2, y_2), (x_3, y_3) is equal to the absolute value of the determinant

$$\begin{vmatrix} x_2 - x_1, & x_3 - x_2 \\ y_2 - y_1, & y_3 - y_2 \end{vmatrix}.$$

Applying this formula to our case, we obtain that the desired area (again to within small quantities of higher order) is equal to the absolute value of the determinant

$$\begin{vmatrix} \frac{\partial x}{\partial \xi} d\xi & \frac{\partial x}{\partial \eta} d\eta \\ \frac{\partial y}{\partial \xi} d\xi & \frac{\partial y}{\partial \eta} d\eta \end{vmatrix} = \frac{D(x, y)}{D(\xi, \eta)} d\xi\, d\eta.$$

Thus

$$\text{the area } P_1P_2P_3P_4 \doteq \left|\frac{D(x, y)}{D(\xi, \eta)}\right| d\xi\, d\eta.$$

Dividing up the figure (Δ) in the plane $\xi\eta$ by straight lines, parallel to the axes, into infinitely small rectangles (and neglecting "irregular" elements on the contour), at the same time we divide up the figure (D) in the xy plane into curvilinear quadrangles of the type under

consideration. Summing the expressions obtained for their areas, we again arrive at formula (10)†.

Ostrogradski's argument as given thus emphasizes an important geometrical idea: *the essence of formula* (10) *lies in the fact that to determine the area of the figure* (*D*) *this figure is divided up not into the traditional rectangular elements, by means of a net of straight lines parallel to the axes, but into curvilinear elements, by means of a net of coordinate lines.*

In some simple cases we can immediately find an expression for "the element of area" in curvilinear coordinates almost without calculation.

For instance, in the case of the transformation to polar coordinates we can reason thus. To the elementary rectangle with sides dr and $d\theta$ in the $r\theta$ plane corresponds the figure in the xy plane bounded by the arcs of the circumferences of radii r and $r+dr$ and the two

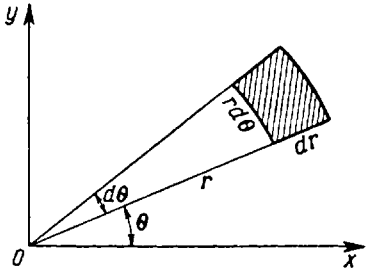

FIG. 42.

rays starting from the origin making angles θ and $\theta+d\theta$ with the x-axis (Fig. 42). Taking this figure as approximately the rectangle with sides dr and $r\,d\theta$, we obtain at once the desired expression $r\,dr\,d\theta$ for the element of area.

356. Change of variables in double integrals. We consider the double integral

$$\iint\limits_{(D)} f(x,y)\,dx\,dy, \tag{13}$$

where the domain (D) is bounded by a simple piece-wise smooth contour (S), and the function $f(x,y)$ is continuous in this domain.

† It is not difficult to make this deduction completely rigorous.

Now we presuppose that the domain (D) is related by means of the formulae (1):

$$x = x(\xi, \eta), \quad y = y(\xi, \eta)$$

to some domain (Δ) in the $\xi\eta$ plane, where all the conditions under which we derived formula (10) in Sec. 353 are satisfied, which expresses the area of the figure (D) in curvilinear coordinates†. By changing the variables in the integral (13), we set out to represent it in the form of an integral extended over the domain (Δ).

To do this we divide up the domain (Δ) by means of a net of piece-wise smooth curves into the sections (Δ_i) $(i = 1, 2, \ldots, n)$; then the domain (D) is divided up by the corresponding (also piece-wise

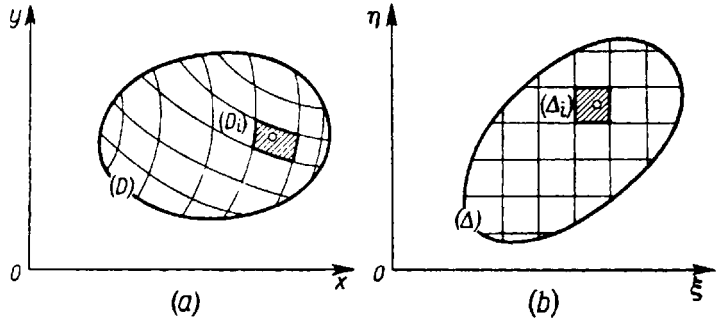

FIG. 43.

smooth) curves into the sections (D_i) (Figs. 43*a*, *b*). In each section (D_i) we choose an arbitrary point (x_i, y_i); finally, we form the integral sum for the integral (13):

$$\sigma = \sum_{i=1}^{n} f(x_i, y_i) D_i,$$

which has this integral as its limit as the greatest of the diameters of the domains (D_i) tends to zero.

Applying formula (11) [Sec. 354] to each domain (D_i), we will have

$$D_i = |J(\bar{\xi}_i, \bar{\eta}_i)| \Delta_i,$$

† We presuppose, thus, the existence and continuity also of the mixed derivatives of the second order $\partial^2 y/\partial\xi\,\partial\eta$ and $\partial^2 y/\partial\eta\,\partial\xi$. Compare the footnote on p. 298.

where $(\bar{\xi}_i, \bar{\eta}_i)$ is some definite point of the domain (Δ_i). Replacing each D_i by this expression in the sum σ, we obtain

$$\sigma = \sum_i f(x_i, y_i)\,|J(\bar{\xi}_i, \bar{\eta}_i)|\,\Delta_i.$$

While the point $(\bar{\xi}_i, \bar{\eta}_i)$ is given by the mean value theorem and we have no freedom in its choice, the point (x_i, y_i) is chosen completely arbitrarily in the domain (D_i). Using this arbitrariness, we set

$$x_i = x(\bar{\xi}_i, \bar{\eta}_i), \quad y_i = y(\bar{\xi}_i, \bar{\eta}_i),$$

i.e. we choose as the point (x_i, y_i) that point of the domain (D_i) which corresponds to the point $(\bar{\xi}_i, \bar{\eta}_i)$ of the domain (Δ_i). Then the sum takes the form

$$\sigma = \sum_i f(x(\bar{\xi}_i, \bar{\eta}_i), y(\bar{\xi}_i, \bar{\eta}_i))\,|J(\bar{\xi}_i, \bar{\eta}_i)|\,\Delta_i;$$

in this form it is clearly the integral sum for the integral

$$\iint_{(\Delta)} f(x(\xi, \eta), y(\xi, \eta))\,|J(\xi, \eta)|\,d\xi\,d\eta. \tag{14}$$

The existence of this integral follows from the fact that the function under the integral is continuous.

If we now make the diameters of all the domains (Δ_i) tend to zero, then by the continuity of the functions (1) the diameters of all the domains (D_i) will also tend to zero. Then the sum σ must tend both to the integral (13) and to the integral (14), because it serves as the integral sum for both of them. Thus,

$$\iint_{(D)} f(x, y)\,dx\,dy = \iint_{(\Delta)} f(x(\xi, \eta), y(\xi, \eta))\,|J(\xi, \eta)|\,d\xi\,d\eta. \tag{15}$$

This formula also solves the question we raised—on the change of variables in a double integral. Formula (10), clearly, is a particular case of it and is obtained from it when $f(x, y) \equiv 1$.

Thus, *in order to make a change of variables in the double integral* (13), *it is necessary not only to substitute in the function f instead of x and y their expressions in* (1), *but also to replace the element of area dx dy by its expression in curvilinear coordinates.*

By means of arguments analogous to those mentioned in Sec. 354, *(4)*, it is also easy to establish here that formula (15) remains valid

in a series of cases when the conditions imposed on the transformation (1) are violated at separate points or along separate lines.

357. The analogy with a simple integral. The integral over an oriented domain. The formula for the change of variables in a double integral is very similar to the formula for the change of variable in a simple definite integral:

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(x(\xi))\,x'(\xi)\,d\xi. \tag{16}$$

However, in formula (16) there is no symbol for the absolute magnitude, which somewhat destroys the analogy. This difference can be easily explained. A simple definite integral is taken along an oriented interval [Sec. 180]: because a may be less or greater than b, equally α may be less or greater than β. At the same time, until now we have only considered a double integral over a non-oriented domain.

It is possible, however, in the case of a double integral also to go on to consider oriented domains. An oriented domain is created by the fact that its contour has attached to it a definite direction for its circuit—positive or negative [Sec. 332]; at the same time the same direction for their circuits is attached to all the closed simple curves within the limits of the domain. If we choose the positive direction for the circuits, then we say that the domain has a positive orientation, and in the opposite case that it has a negative orientation.

It is natural to agree to take as the area for the oriented domain (D) its usual area with the plus sign, if the domain has a positive orientation, and with the minus sign in the opposite case. When the domain (D) is divided up into the sections (D_i) these sections, as we indicated, are oriented according to the orientation of the whole domain; their areas also have the corresponding signs.

Now, after the pattern of Sec. 338 we can form the concept of a double integral

$$\iint_{(D)} f(x, y)\,dx\,dy$$

for the oriented domain (D), where this integral coincides with that defined earlier, if the domain has the positive orientation, and differs from it in sign in the case of the negative orientation.

This new point of view on double integrals enables us first of all to rewrite formula (10), Sec. 353, which expresses the area in curvilinear coordinates, without the sign for the absolute value of the functional determinant:

$$D = \iint_{(\Delta)} \frac{D(x, y)}{D(\xi, \eta)}\,d\xi\,d\eta = \iint_{(\Delta)} J(\xi, \eta)\,d\xi\,d\eta,$$

provided only that the orientations of the domains (D) and (Δ) are made consistent. This follows immediately from the remark in Sec. 354, *(1)*.

For the same case the conditions of formula (11), Sec. 354 can also be rewritten without the sign for the absolute value:

$$D = J(\bar{\xi}, \bar{\eta})\,\Delta,$$

and in this form it is a natural generalization of Lagrange's formula.

Finally, now the general formula (15) can also be rewritten for consistently oriented domains (D) and (Δ) in the form

$$\iint\limits_{(D)} f(x, y)\,dx\,dy = \iint\limits_{(\Delta)} f(x(\xi, \eta), y(\xi, \eta))\,J(\xi, \eta)\,d\xi\,d\eta.$$

Thus, it is only necessary to set down the simple and double integrals under identical conditions for the analogy to be complete.

However, in what is presented below we nevertheless return to the usual point of view and we will consider double integrals extended over a non-oriented domain.

358. Examples. *(1)* We will evaluate the area of the figure bounded by the *lemnisca* (Fig. 44):

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2).$$

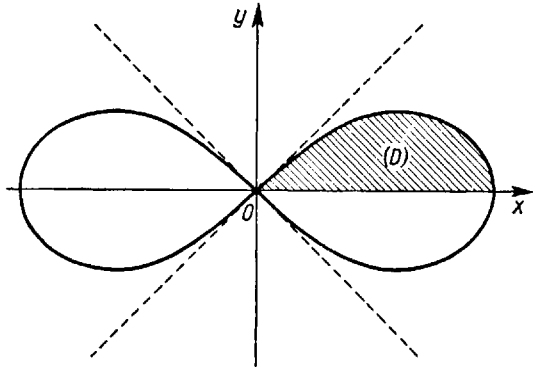

FIG. 44.

The presence of the binomial $x^2 + y^2$ suggests the transition to polar coordinates, by letting

$$x = r\cos\theta, \quad y = r\sin\theta$$

and evaluating the desired area by means of the formula

$$D = \iint\limits_{(\Delta)} r\,dr\,d\theta. \tag{17}$$

Because our curve is symmetric with respect to the coordinate axes (this is easy to verify from the equation of the curve, because its form is unaltered when x is replaced by $-x$ or y by $-y$), then it is sufficient to determine the area of the portion (D) of the figure, contained in the first quadrant, and then multiply it by four.

The polar equation of the lemnisca is

$$r^2 = 2a^2 \cos 2\theta,$$

where (if we limit ourselves to the first quadrant) θ only needs to vary from 0 to $\pi/4$ in view of the fact that $\cos 2\theta$ must be positive. Thus, the domain (Δ) in the $r\theta$ plane, corresponding to (D), is bounded by the curve

$$r = a\sqrt{(2\cos 2\theta)}$$

(the image of the lemnisca), the segment of the r-axis (which corresponds to the segment of the x-axis) and the segment of the θ-axis from $\theta = 0$ to $\theta = \pi/4$ (the image of one only of the points at the origin—where the one-one correspondence is violated†).

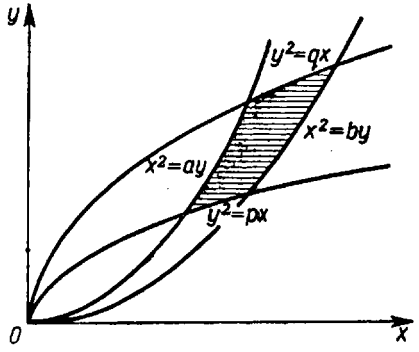

FIG. 45.

We have

$$D = \int_0^{\pi/4} d\theta \int_0^{a\sqrt{(2\cos 2\theta)}} r\,dr = a^2 \int_0^{\pi/4} \cos 2\theta\, d\theta = \frac{a^2}{2},$$

so that the whole required area is $2a^2$.

(2) Now we indicate another approach to the choice of a system of curvilinear coordinates which is often useful in determining the area of a curvilinear quadrangle. If both pairs of curves representing opposite sides of this quadrangle are part, in turn, of an actual family of curves which cover the plane and depend on one parameter, then these two families can in fact be taken quite naturally as the net of coordinate lines. Their parameters usually give a convenient system of curvilinear coordinates for the given case.

We will explain this method by means of an example. Suppose that it is required to find the area of the figure bounded by the parabolas

$$y^2 = px,\ y^2 = qx,\ x^2 = ay,\ x^2 = by,$$

where $0 < p < q$ and $0 < a < b$ (Fig. 45).

† See, in this connection, *(4)* in Sec. 354.

Here it is convenient to consider two families of parabolas:

$$y^2 = \xi x \qquad (p \leqslant \xi \leqslant q)$$

and

$$x^2 = \eta y \qquad (a \leqslant \eta \leqslant b),$$

each of which covers our figure, and we form a net of coordinate lines from them. This is equivalent to our taking their parameters ξ and η as curvilinear coordinates. We know all this by Sec. 352, *(2)*; from the equations we have written down we have: $x = \sqrt[3]{(\xi\eta^2)}$ and $y = \sqrt[3]{(\xi^2\eta)}$, so that the functional determinant is

$$J = -\tfrac{1}{3}.$$

Hence we at once obtain

$$D = \tfrac{1}{3}(q-p)(b-a).$$

(3) To find the volume V of the body cut out by the cylinder $x^2 + y^2 = Rx$ from the sphere $x^2 + y^2 + z^2 \leqslant R^2$ (Fig. 46) (this body is sometimes called "Viviani's solid"—named after an Italian mathematician of the seventeenth century who was first interested in it).

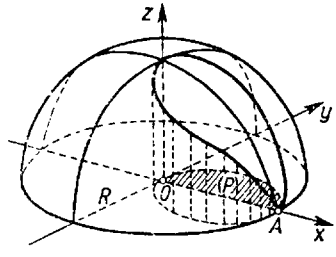

FIG. 46.

We have [by formula (2), Sec. 336]:

$$V = 4\iint\limits_{(P)} V(R^2 - x^2 - y^2)\,dx\,dy,$$

where P is the semicircle in the first quadrant of the xy plane bounded by the lines $x = 0$ and $x^2 + y^2 = Rx$.

We can evaluate this integral immediately by building it up from elementary sections; however, it is considerably simplified by a transformation to polar coordinates.

The polar equation of the contour (P), i.e. the semicircle, will be $r = R\cos\theta$ as θ varies from 0 to $\pi/2$. Thus,

$$V = 4\int_0^{\pi/2} d\theta \int_0^{R\cos\theta} V(R^2 - r^2)\,r\,dr = \frac{4}{3}R^3\int_0^{\pi/2}(1 - \sin^3\theta)\,d\theta = \frac{4}{3}R^3\left(\frac{\pi}{2} - \frac{2}{3}\right).$$

(4) Polar coordinates are often useful in establishing the existence of "improper" double integrals [Sec. 344, *(4)* and the Remark]. By way of an example we establish the conditions for the existence of the integrals $(m>0)$;

(a) $\iint\limits_{x^2+y^2\leqslant 1} \dfrac{dx\,dy}{(x^2+y^2)^m}$ (singular point at the origin);

(b) $\iint\limits_{x^2+y^2\leqslant 1} \dfrac{dx\,dy}{(1-x^2-y^2)^m}$ (singular line—the unit circle around the origin);

(c) $\iint\limits_{x^2+y^2\geqslant 1} \dfrac{dx\,dy}{(x^2+y^2)^m}$ (extending over an infinite domain).

In case (a) we have

$$\iint\limits_{0<\varepsilon^2\leqslant x^2+y^2\leqslant 1} \frac{dx\,dy}{(x^2+y^2)^m} = 2\pi \int\limits_{\varepsilon}^{1} \frac{dr}{r^{2m-1}},$$

which, as $\varepsilon \to 0$, tends to the finite limit $2\pi \int\limits_0^1 dr/r^{2m-1}$, provided only that $2m-1<1$, i.e. when $m<1$. In case (b):

$$\iint\limits_{x^2+y^2\leqslant (1-\varepsilon)^2} \frac{dx\,dy}{(1-x^2-y^2)^m} = 2\pi \int\limits_0^{1-\varepsilon} \frac{r\,dr}{(1-r^2)^m} \to 2\pi \int\limits_0^1 \frac{r\,dr}{(1-r^2)^m} \quad (\text{as } \varepsilon \to 0),$$

if $m<1$. Finally, for the integral (c):

$$\iint\limits_{1\leqslant x^2+y^2\leqslant R^2} \frac{dx\,dy}{(x^2+y^2)^m} = 2\pi \int\limits_1^R \frac{dr}{r^{2m-1}} \to 2\pi \int\limits_1^\infty \frac{dr}{r^{2m-1}} \quad (\text{as } R \to \infty),$$

if $2m-1>1$, i.e. $m>1$.

359. Historical note. Double integrals were first introduced by Euler—in a work presented to the academy of St. Petersburg in 1769. First he considered an indefinite double integral of this kind $\int\int Z\,dx\,dy$ as a function of x and y, which reduced to the expression $Z\,dx\,dy$ if it were successively differentiated with respect to these variables in one order or the other. Thus, this double integral is identified with both the repeated integrals

$$\int dx \int Z\,dy \quad \text{and} \quad \int dy \int Z\,dx,$$

and in general its expression keeps the form of summands both of an arbitrary function only of x, and of an arbitrary function only of y.

Then, in connection with the problem of calculating the volume and surface of a solid, Euler introduced the definite double integral, treating it at the same time both as the sum of its elements, and as a repeated integral—of either one

type or the other. If the first to be evaluated is, say, $\int Z\,dy$ (for constant x), then it is extended over all values of y between limits which are, generally speaking, independent of x. As a result we obtain a function of x alone (and not of x and y, as in the case of an indefinite integral), which is also integrated between constant limits†. All the limits are defined by the "basis" under consideration (this is what Euler called the domain of integration: in fact, it usually is the base for the solid whose volume and surface are sought).

In the work mentioned above Euler also examined the question of the change of the variables x, y under the sign of the double integral to new variables t, u. Here it is presupposed that x, y are functions of t, u, where

$$dx = R\,dt + S\,du, \quad dy = T\,dt + V\,du, \tag{18}$$

so that R, S, T, V denote the corresponding partial derivatives. The limits between which the new variables vary are defined by the previous "basis". After we have substituted in the function under the integral, instead of x and y, their expressions in terms of t and u, there remains open only the question: what do we substitute for the element of area $dx\,dy$?

Euler at once rejected the idea of simply substituting for dx and dy their expressions in (18). He correctly remarked, further, that in general "*there is no reasonable basis for supposing that the expression introduced into the calculation in place of dx dy should be equal to it*": it is important only that the final results of the integration should coincide. In addition this expression must contain the factor $dt\,du$, i.e. it must have the form $Z\,dt\,du$, so that the integral with respect to the new variables can be treated as a repeated integral.

In order to find the factor Z Euler used the successive introduction of the new variables, so that at each stage he only needed to change one variable in a simple integral (the inner integral). As a result he arrived at the expression

$$Z = ST - RV,$$

in which the reader will easily recognize the functional determinant. However, Euler had already begun to have a "serious doubt", in connection with the fact that, if he interchanged x and y, it was possible, completely analogously, to obtain the expression

$$Z = RV - ST$$

with the opposite sign. This is explained by the fact that nowhere here did Euler write out the limits of the integral or give any attention to their order. The difficulty is resolved by considering that the area must always be obtained as a positive number, and this means that Z must be taken as positive. Thus, finally, in place of $dx\,dy$ we must substitute the expression

$$|ST - RV|\,dt\,du. \tag{19}$$

Euler obtained this expression by means of purely formal operations, by no means equating this to the element of area $dx\,dy$, and also in general not clarifying its geometrical meaning.

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† In this work Euler does not set limits for the integrals.

Some years later Lagrange drew Euler's attention to the same problem of the change of variables, but this time in a triple integral. If we rephrase his discoveries for the case of a double integral, we can say that Lagrange attempted to prove them as if the elementary area $dx\,dy$ were simply equal to the expression (19).

Only in 1836 in an address to the academy of St. Petersburg, entitled *Transformation of Variables in Multiple Integrals*, did Ostrogradski completely clarify the question. First of all—by a simple example on the transformation of variables to polar coordinates—he established the invalidity of Lagrange's arguments. Then he gave an original treatment of the problem, essentially that given above in Sec. 355. Expression (19) gained a clear geometrical significance, as the area of an elementary curvilinear quadrangle†, where for the first time a convincing reason is given for the necessity of considering the absolute value of the determinant.

† Which need not at all be equal to the elementary rectangle.

CHAPTER 22

THE AREA OF A SURFACE. SURFACE INTEGRALS

§ 1. Two-sided surfaces

360. Parametric representation of a surface. Now we turn to the problem of the analytical representation of a surface in space [cf. Sec. 213], starting with a very important type—not yet introduced to the reader—of such a representation, called a *parametric* representation.

In Sec. 212 we have already discussed the parametric representation of a curve in space [see (15) in that section]

$$x = \varphi(t), \quad y = \psi(t), \quad z = \chi(t), \tag{1}$$

so that the position of a point on it was defined by the value of one parameter t, varying in some interval. To define the position of a point on a surface given by an explicit equation, say,

$$z = f(x, y), \tag{2}$$

we already have two parameters to deal with, their part being played here by the abscissa x and the ordinate y. In the general case two arbitrary variables u and v appear as the parameters, and *the parametric representation of the surface* is realized by means of three equations

$$x = \varphi(u, v), \quad y = \psi(u, v), \quad z = \chi(u, v)^{\dagger}, \tag{3}$$

where the functions φ, ψ, χ are defined and continuous in some domain (Δ) in the "plane of the parameters" uv.

† This way of giving a surface by means of two parameters, first introduced by Euler, was especially widely and profitably used in differential geometry by Gauss.

Our fundamental case will be that when every point of the surface is obtained from only one pair of values of the parameters, so that equation (3) establishes a one–one correspondence between the points of the surface and the points of the plane domain (Δ); we will call such a surface *simple*. Here we will presuppose that the domain (Δ) is also a surface bounded by simple closed contours; they are necessarily related to one another by formulae (3).

The parameters u and v are called the *curvilinear coordinates* of the corresponding point. If we fix the value of one of the curvilinear coordinates in equations (3), for instance if we set $u = u_0$, then, clearly, we obtain the equation of some curve

$$x = \varphi(u_0, v), \quad y = \psi(u_0, v), \quad z = \chi(u_0, v),$$

all of whose points lie on the surface. By changing the value of u_0, we obtain a whole family of such "(u) curves". Analogously, by fixing the value of $v = v_0$, we also obtain a curve on our surface

$$x = \varphi(u, v_0), \quad y = \psi(u, v_0), \quad z = \chi(u, v_0);$$

we can also construct a whole family of such "(v) curves". All such lines are called coordinate lines of the surface. If the surface is simple, then through any point of it passes one coordinate line of each family.

All this is already known to the reader, but only for the case when the surface is plane. [Cf. Sec. 352.]

Now we presuppose that the functions (3) are not only continuous, but also have continuous partial derivatives of the first order in the domain (Δ), and we consider the functional matrix

$$\begin{pmatrix} \varphi'_u & \psi'_u & \chi'_u \\ \varphi'_v & \psi'_v & \chi'_v \end{pmatrix}. \tag{4}$$

For the values (u_0, v_0) of the parameters, defining the point $M_0(x_0, y_0, z_0)$ of the surface, let at least one of the second order determinants from the matrix (4) be distinct from zero, for instance, let

$$\begin{vmatrix} \varphi'_u & \psi'_u \\ \varphi'_v & \psi'_v \end{vmatrix} \neq 0.$$

Then, rewriting the first two of equations (3) in the form

$$\varphi(u, v) - x = 0, \quad \psi(u, v) - y = 0,$$

on the basis of Theorem 3, Sec. 317 we can assert that this system of two equations with four variables u, v, x, y (if we limit ourselves to their values which are near to the values which interest us: u_0, v_0, x_0, y_0) defines the variables u, v as single-valued functions of x, y: $u = u(x, y)$, $v = v(x, y)$, which are continuous together with their derivatives. Finally, by substituting these expressions for u, v in the third of the equations (3), we arrive at a representation of the part of the surface surrounding the point M_0 by means of an explicit equation

$$z = \chi(u(x, y), v(x, y)) = f(x, y)$$

of type (2), where the function f is also continuous and has continuous derivatives.

Only when all three determinants of the matrix (4) vanish simultaneously (the corresponding point M_0 on the surface in this case is called singular) is it impossible to find such a representation.

Let us take on a simple surface (3) some non-singular point $M(x, y, z)$. Then, as we have only just seen, in some neighbourhood of it the surface is represented by an explicit equation of one form or the other and, hence [Sec. 213], at the point M it has a tangent plane. The last equation can be written in the form

$$A(X-x) + B(Y-y) + C(Z-z) = 0\dagger, \tag{5}$$

where the coefficients A, B, C have yet to be defined.

If in the equations of the surface we assign to v the value corresponding to the chosen point M, then we obtain the equation of the "(v) curve" passing through this point. The tangent to this curve at the point M is given by the equations [Sec. 212, (16)]

$$\frac{X-x}{x'_u} = \frac{Y-y}{y'_u} = \frac{Z-z}{z'_u}.$$

Analogously, by fixing u, we obtain another coordinate line passing through the point M ("a (u) curve"), which has the tangent

$$\frac{X-x}{x'_v} = \frac{Y-y}{y'_v} = \frac{Z-z}{z'_v}$$

at this point.

† As usual, we denote by X, Y, Z the current coordinates, as distinct from the coordinates x, y, z of a fixed point on the surface.

Because both of these tangents must lie in the plane (5), then the conditions

$$Ax'_u + By'_u + Cz'_u = 0,$$
$$Ax'_v + By'_v + Cz'_v = 0$$

are satisfied. In this case the coordinates A, B, C must be proportional to the determinants of the matrix

$$\begin{pmatrix} x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \end{pmatrix}^{\dagger}. \tag{4a}$$

Usually we can suppose that they are simply equal to these determinants, so that from now on we will always assume that

$$A = \begin{vmatrix} y'_u & z'_u \\ y'_v & z'_v \end{vmatrix}, \quad B = \begin{vmatrix} z'_u & x'_u \\ z'_v & x'_v \end{vmatrix}, \quad C = \begin{vmatrix} x'_u & y'_u \\ x'_v & y'_v \end{vmatrix}. \tag{6}$$

We note also that the direction cosines of the normal will be

$$\left.\begin{aligned} \cos\lambda = \frac{A}{\pm\sqrt{(A^2+B^2+C^2)}}, \quad \cos\mu = \frac{B}{\pm\sqrt{(A^2+B^2+C^2)}}, \\ \cos\nu = \frac{C}{\pm\sqrt{(A^2+B^2+C^2)}}. \end{aligned}\right\} \tag{7}$$

We have limited ourselves until now to the consideration of simple curves. In the general case we will presuppose that we are dealing both with surfaces bounded by contours, and also with closed surfaces consisting of a finite number of simple sections, adjoining one another, but not crossing. If we succeed in representing such a surface "of general form" by mean of the parametric equations (3), then it is known that the one–one correspondence between its points and the points of the plane domain (Δ) is violated.

Examples. (1) We will consider the sphere of radius R with centre at the origin (Fig. 47): $x^2 + y^2 + z^2 = R^2$. As we wish to obtain the usual parametric representation for it, we draw the "equatorial" section AKA', and the "meridian" $PMKP'$ through the "poles" P, P' and the point M under consideration. The position of the point M on the sphere can be defined by the angles $\varphi = \sphericalangle POM$ and $\theta = \sphericalangle AOK$. We have $z = NM = R\cos\varphi$. Then, $ON = R\sin\varphi$, and on

† This matrix only differs from the matrix (4) in notation. We note that at the point under consideration its determinants are not all zero.

ON the coordinates x and y (the same for M as for N) are given by $x = ON\cos\theta$, $y = ON\sin\theta$. Finally, *the parametric equations of the sphere are obtained in the form:*

$$x = R\sin\varphi\cos\theta, \quad y = R\sin\varphi\sin\theta, \quad z = R\cos\varphi,$$

where the angle φ need only vary from 0 to π, and the angle θ from 0 to 2π.

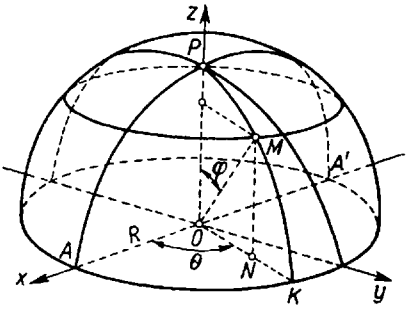

FIG. 47.

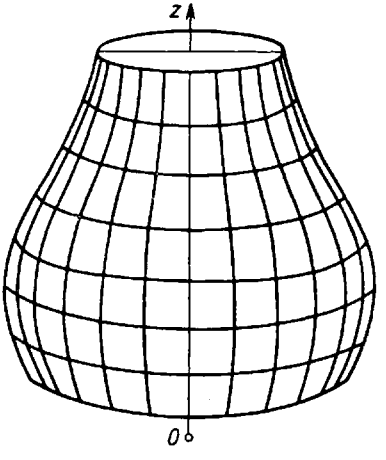

FIG. 48.

However, the correspondence between the points on the spherical surface and the points of the rectangle $[0, \pi; 0, 2\pi]$ on the $\varphi\theta$ plane will not be one-one: not only do the values $\theta = 0$ and $\theta = 2\pi$ give the same point of the surface, but when $\varphi = 0$ (π), whatever may be the value of θ, only one point is obtained—the pole $P(P')$.

It is easy to see that one family of coordinate lines on the sphere consists of the meridians (θ = const), and the other of the parallel circles (φ = const).

(2) We can generalize the preceding example in the following way. In the xz plane let a ("generating") curve be given by means of its parametric equations

$$x = \varphi(u), \quad z = \psi(u) \quad (\alpha \leqslant u \leqslant \beta),$$

where $\varphi(u) \geqslant 0$. We begin by rotating it, to form a solid figure, about the z-axis (Fig. 48). If we denote the angle of rotation by v, then the equations obtained for the *surface of rotation* can be written in the form

$$x = \varphi(u)\cos v, \quad y = \varphi(u)\sin v, \quad z = \psi(u) \quad (\alpha \leqslant u \leqslant \beta, 0 \leqslant v \leqslant 2\pi).$$

If in the xz plane we take the semicircle

$$x = R\sin u, \quad z = R\cos u \quad (0 \leqslant u \leqslant \pi)$$

and rotate it about the z-axis, then we obtain the parametric representation of the *sphere* generated in this way in the previous form (to within notation).

The coordinate lines here also are the different positions of the generator (the meridians) and the parallel circles.

361. The side of a surface. We will establish the concept of the side of a surface, which is important in what follows.

In a number of cases this concept is intuitively clear. If a surface is given by an explicit equation of the form $z = f(x, y)$, we can speak of the upper side or the lower side of the surface†. If the surface bounds some solid, then it is also easy to represent two sides of it—the inside, which is turned towards the solid and the outside, which is turned towards the space surrounding the body.

Starting from this intuitive representation, we will now try to give a precise definition of the concept of a side of a surface.

We consider a surface (S), closed or not, and we presuppose that at each point of the surface there is a definite tangent plane whose position varies continuously together with the point of contact.

Taking a definite point M on the surface, we draw the normal at this point, to which we assign a definite direction—one of two possibilities (they differ from one another by the signs of their direction cosines). We draw a closed contour on the surface, starting from M_0 and returning to M_0, where we presuppose that it does not cross the boundary of the surface (if it has one). We make the point M go around this contour and at each of its successive positions we will draw the normal in that one of the two directions from which

† We will often use such an expression (implying in this that the z-axis itself is directed vertically upwards).

it passes continuously to the direction chosen by us at the original position M_0. Here one of two things can happen: after going round the contour either we return to the point M_0 with the same direction of the normal, or with the direction opposite to the original direction.

If for any point M_0 and any contour M_0AM_0 passing through it the latter situation occurs, then for any other point M_1 it is easy to construct a closed contour which, starting from M_1 and returning to it, brings us back to this point with the direction of the normal opposite to the original direction. Thus, for instance, we can take the contour $M_1M_0AM_0M_1$, if by M_1M_0 we understand any curve lying on the surface and joining M_1 and M_0, but not crossing the boundary of the surface, and by M_0M_1 the same curve in the opposite direction.

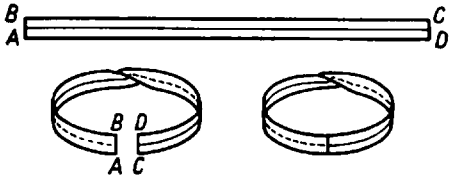

FIG. 49.

In this case the surface is said to be *one-sided.* The classical example of such a surface is the so-called Möbius band (Fig. 49). We can construct a model of this if we twist a rectangular piece of paper $ABCD$ once, and paste it together so that the point A coincides with C, and B with D. If we begin painting the twisted ring obtained with any colour, then, without crossing its boundary, we can colour the whole ring with this colour. From now on we will exclude such surfaces from those we are considering.

Now we presuppose that for any point M_0, and any closed contour passing through M_0 and not crossing the boundary of the surface, after going round the contour we invariably return to the original point M_0 with the original direction of the normal. Under these conditions the surface is said to be *two-sided.*

Let S be a two-sided surface. We will take any point M_0 on it and we will draw the normal at this point in a definite direction. Taking any other point M_1 of the surface, we join M_0 and M_1 by

an arbitrary path (K) lying on the surface and not crossing its boundary, and we make the point M go from M_0 to M_1 along this path. If as we do this the direction of the normal varies continuously, then the point M arrives at the position M_1 with the direction of the normal completely determined, independently of the choice of the path (K). In fact, if, in going from the point M_0 to the point M_1 along two distinct paths (K_1) and (K_2), we were to obtain two different directions of the normal at the point M_1, then the closed path $M_0(K_1)M_1(K_2^{-1})M_0$† would bring us to the point M_0 with the direction of the normal opposite to the original direction, which would contradict the definition of a two-sided surface.

Thus, on a two-sided surface the choice of the direction of the normal at one point defines uniquely the choice of the direction of the normal at all points of the surface. *The totality of all points of a surface with the directions of their normals assigned by the given rule is said to form a definite side of the surface.*

Examples. (1) The simplest and most important example of a two-sided surface is the surface expressed by means of the explicit equation $z = z(x, y)$, with the hypothesis that the function z is continuous in some plane domain (D) and admits continuous partial derivatives

$$p = \frac{\partial z}{\partial x} \quad \text{and} \quad q = \frac{\partial z}{\partial y}.$$

In this case the direction cosines of the normal to the surface can be expressed [Sec. 213, (24)]:

$$\left.\begin{aligned} \cos\lambda &= \frac{-p}{\pm\sqrt{(1+p^2+q^2)}}, \quad \cos\mu = \frac{-q}{\pm\sqrt{(1+p^2+q^2)}}, \\ \cos\nu &= \frac{1}{\pm\sqrt{(1+p^2+q^2)}}. \end{aligned}\right\} \tag{8}$$

By choosing a definite sign before the radical we establish a definite direction of the normal at all points of the surface. Because the direction cosines, by virtue of our hypotheses, will be continuous functions of the coordinates of the point, then the established direction of the normal will also depend continuously on the position of the point. Hence it is clear that *the choice of the sign before the radical in the formulae for* $\cos\lambda$, $\cos\mu$, $\cos\nu$ *defines a side of the surface* in precisely the same sense as was given above to this concept.

† By (K_2^{-1}) we understand the curve (K_2), but drawn in the opposite direction—from M_1 to M_0.

If we choose the plus sign before the radical, then at all points of the surface

$$\cos\nu = \frac{1}{\sqrt{(1+p^2+q^2)}}$$

will be positive, i.e. the angle made by the normal with the z-axis, corresponding to this choice of side, will be acute. Thus, the side of the surface defined by the given choice of sign is the inner side. On the other hand, the choice of the minus sign in the expressions for the direction cosines of the normal characterizes the outer side of the surface: the normals make obtuse angles with the z-axis.

(2) Now we consider a more generally arbitrary simple surface (S), given by the parametric equations

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \tag{3a}$$

where the parameters u, v vary in some bounded domain (Δ) on the uv plane. We presuppose that the surface is *smooth*: this means that the functions (3a) are continuous in (Δ) together with their partial derivatives and that there are no singular points on the surface.

For the direction cosines of the normal to the surface we had the formulae (7). And in this case *the choice of the sign before the radical characterizes the side of the surface*, so that the surface is two-sided.

In fact, once the sign is chosen, the formulae (7) assign to each point of the surface (because it corresponds to only one pair of values u, v) one definite direction of the normal, which varies continuously as the point moves.

If the hypothesis that the surface is simple is violated, it is impossible to assert without reservations that it is two-sided. In this case we can find a point M_0 on the surface which corresponds to two distinct pairs $u_0 v_0$ and $u_1 v_1$ of values of the parameters, and it may happen that for these values the formulae (7), even if the sign before the radical is chosen uniquely, may define normals in opposite directions at the point M_0. If this is in fact the case, then the surface will be one-sided.

Remark. If the surface (S) is closed and bounds some solid, then [Sec. 360], although it can be expressed by equations (3a), yet on this occasion the hypothesis that the correspondence between the points of the surface and the points of the plane domain (Δ) is one–one is not fully realized. Nevertheless, because it is immediately clear that the surface has two sides—an inside and an outside—in this case also *the choice of the sign in the formulae* (7) *defines a side of the surface.*

362. The orientation of a surface and the choice of a side of it. Let (S) be a simple smooth surface, bounded by a simple contour (L); we choose a definite side of this surface. We take a definite direction of the circuit of the contour (L) as the positive direction by the following rule: the circuit must be counterclockwise to an observer moving in this direction along the contour in such a way

that the normal to the surface, corresponding to the chosen side, passes through him from his feet to his head. The words "in a counterclockwise direction" mean, to be more precise, that the observer must see the part of the surface immediately adjacent to him on his left. By the same rule we can simultaneously establish the positive direction of the circuit for every simple closed contour lying on the surface and bounding some part of it†. The direction of the circuit opposite to the positive direction we will call *negative.*

Together, all this forms the basis for the concept of the orientation of a surface.

If we start from the other side of the surface, then the normals reverse their direction, the position of the observer is changed, in connection with which, by our rule, it is necessary to transpose the positive and negative directions of the circuit of the contour (L) and the other curves lying on the surface: the surface changes its orientation. Thus, if we always keep to the rule we have established, *the choice of the side of a surface determines its orientation* and, conversely, *the choice of a positive direction for the circuit of a contour of a surface uniquely defines a side of it.*

Remark. At the basis of this rule lies the idea of a rotation in a counterclockwise direction, and *to avoid confusion we will always start from this idea.*

We will connect with this the actual ordering of the coordinate axes in space. In fact, in all problems for which this has any significance *we will use the so-called right coordinate system* (Fig. 50*a*): in it the axes are ordered in such a way that the rotation from the x-axis to the y-axis (through an angle of $+\pi/2$), if we look at it from the side of the positive part of the z-axis, seems to go in a counterclockwise direction†. (In Fig. 50*b* a system of axes is illustrated for which the given rotation is clockwise‡: such a system is called left-handed.)

Now we give an application of the ideas presented above, on the connection between the choice of a side of a surface and the

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† It is only necessary to take this part into consideration in defining the positive direction on the contour.

‡ This remains true under a cyclic permutation of the letters xyz.

creation of a definite orientation on it, to the problem of the choice of the sign in the formulae (7) for the direction cosines of the normal to the curve.

We shall return to the surface (S) considered in the first section, assuming that we have chosen a definite side on it, and with it an orientation. A contour (L) of our surface corresponds to a contour (Λ) of the domain (Δ) on the "plane of parameters" uv. Let us suppose (it is always easy to do this) that to the positive circuit

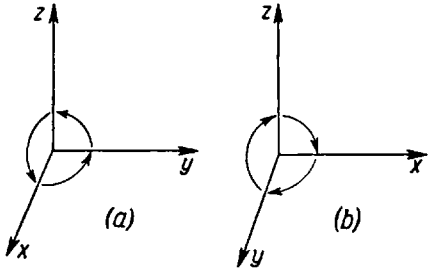

FIG. 50.

of the contour (Λ) corresponds the positive circuit of the contour (L). Then for any contours (λ) in the domain (Δ) and (l) on the surface (S) corresponding to one another the same thing holds: the positive circuit of (λ) implies the positive circuit of (l)†.

Under these conditions, in order to characterize the chosen side of the surface we must take the plus sign before the radical in the formulae (7) *for the direction cosines of the normal.*

To prove this it is sufficient to establish that at least at one point the direction defined by these formulae with the plus sign coincides with the necessary direction of the normal. We will take any interior point M_0 on the surface; the point $m_0(u_0, v_0)$ in the domain (Δ) corresponds to it. At this point let, say, the determinant

$$C = \begin{vmatrix} x'_u & y'_u \\ x'_v & y'_v \end{vmatrix}$$

† Because one can judge the direction of the circuit of a contour by the direction in which any part of it is described, then the given assertion is obvious for a contour (λ) having a common part with (Λ), and then it is easily extended to the general case.

be distinct from zero. Then we can find so small a neighbourhood of the point m_0 in the uv plane, bounded by the contour (λ), that the neighbourhood of the point M_0 on the surface (S) corresponding to it and bounded by the contour (l) can be represented by an explicit equation of the form $z = f(x, y)$ [Sec. 360] and, hence, the projection on the xy plane is one-one. We denote the contour of this projection on the xy plane by (k) (Fig. 51).

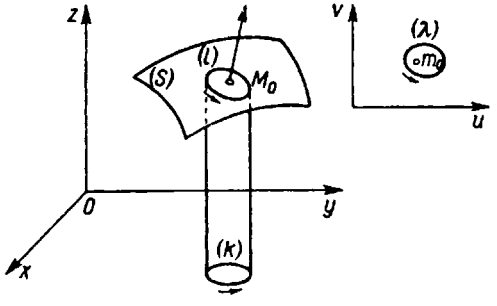

FIG. 51.

If at the point under consideration and in its neighbourhood the determinant $C > 0$, then to the positive circuit of the contour (λ) corresponds the positive circuit (i.e. with the chosen ordering of the axes, the circuit is counterclockwise) of the contour (k) [see Sec. 354, *(1)*]. As we can see from the figure, for the circuit of the contour (l) on the surface, which corresponds to this, to be counterclockwise, we need to look at it from above, so that in this case the normal at the point M_0 must be directed upwards, i.e. it must make an acute angle with the z-axis. This in fact also holds by formulae (7), if we take the plus sign in it, because when $C > 0$ $\cos\nu > 0$ also. However, when $C < 0$ the normal must make an obtuse angle with the z-axis, which is in fact also realized with the given choice of sign, because when $C < 0$, $\cos\nu < 0$ also.

363. The case of a piece-wise smooth surface. The ideas developed in Sec. 362 also provide a convenient means for the extension of the concept of a side of a surface to the general case of a surface consisting of several simple and smooth pieces; for brevity we will call such a surface *piece-wise smooth*. The considerations presented

in Sec. 361 are not immediately applicable in this case, because along the "edges", joining the given pieces of the surface, a definite tangent plane does not in general exist, and in passing over them it is not possible to speak of the continuous variation in the direction of the normal.

Let there be given a surface (S), consisting of the simple and smooth pieces (S_1), (S_2), ..., joining each other along edges, in general part of their contours. We presuppose, first of all, that *each of these pieces separately is a two-sided surface.* But this, of course, is not sufficient for us to be able to consider the whole surface (S) as two-sided; in fact, a Mobius surface can easily be constructed from two simple smooth two-sided pieces.

On the contour (K_i) of each piece (S_i) $(i = 1, 2, ...)$ we choose as positive one of the two directions; in this way, as we have seen, we fix the side of the surface (S_i). If this choice can always be made in such a way that *the common part of the contours*† *of two adjoining*

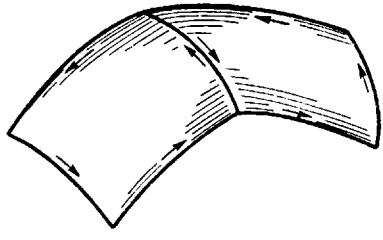

FIG. 52.

pieces is described in opposite directions for these pieces (Fig. 52), only then is the surface (S) two-sided. A side of the surface (S) is defined as the totality of the sides of its parts chosen in the given way.

If we reverse the direction of at least one circuit of a contour, then for our conditions to be observed we must do the same with all the contours. Then the chosen sides of all the pieces (S_i) are replaced by those opposite to them; their totality forms the second side of the surface.

† This part may consist of separate pieces.

The considerations in Sec. 362 regarding the connection between the sides of a surface and its orientation still hold in the general case of a surface which has a contour.

§ 2. The area of a curved surface

364. Schwarz's example. The concept of the area of a curved surface has a well-known analogy with the concept of the length of a curved line. The length of a (non-closed) arc we define as the limit of the perimeter of the broken line inscribed in the arc—under the condition that the lengths of all its sides tend to zero. In the case of a curved surface (also, say, non-closed) it would be natural to consider a many-sided surface inscribed in it and to define the area of the curved surface as the limit of the area of this many-sided surface—under the condition that the diameters of all its sides tend to zero.

At the end of the last century, however, *the unsuitability of this definition was revealed.* In fact, in 1883 Schwarz showed that the given limit does not even exist for the simple case of the surface of a right curved cylinder. We will quote this instructive example.

Let there be given such a cylinder of radius R and height H. We inscribe in it a many-sided surface in the following way. Dividing up the height of the cylinder into m equal parts, we draw through the points of division planes which are perpendicular to the axis of the cylinder, so that we obtain $m+1$ circles on its surface (excluding from this the circles formed by both ends of the cylinder). We divide up each of these circles into n equal parts so that the points of division which fall outside a circle are situated over the centres of the arcs of the circle lying below.

We take, further, the triangles formed by the chords of all these arcs and the segments joining the end-points of the chords and those points of division of the circles lying above and below, which are situated precisely above or below the centres of the corresponding arcs (Fig. 53). Taken together these $2mn$ equal triangles form our required many-sided surface $\left(\sum_{m,n}\right)$; a model of it is represented in Fig. 54.

Now we calculate the area σ of each of these triangles. As our basis we will take a chord, the length of which is equal to

$$2R\sin\frac{\pi}{n}.$$

To find the height AB of a triangle (see diagram) we note that

$$AB = \sqrt{(AC^2 + BC^2)},$$

where

$$AC = OC - OA = R\left(1 - \cos\frac{\pi}{n}\right), \quad BC = \frac{H}{m}.$$

Thus, the area of one of the triangles is equal to

$$\sigma = R\sin\frac{\pi}{n}\sqrt{\left[R^2\left(1 - \cos\frac{\pi}{n}\right)^2 + \left(\frac{H}{m}\right)^2\right]},$$

and the area of the whole many-sided surface will be

$$\sum_{m,n} = 2mn\sigma = 2Rn\sin\frac{\pi}{n}\sqrt{\left[R^2m^2\left(1 - \cos\frac{\pi}{n}\right)^2 + H^2\right]}.$$

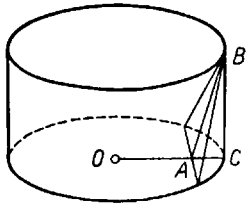

FIG. 53.

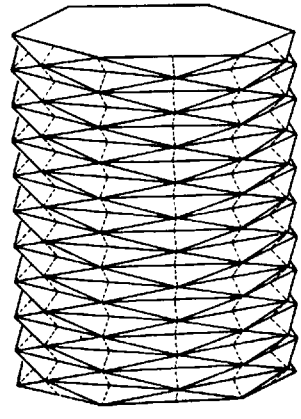

FIG. 54.

When m and n increase without limit, then the diameters of all the triangles tend to zero, but the area $\sum_{m,n}$ does not have a limit. In fact, let us suppose that m and n increase in such a way that the ratio m/n^2 tends to a definite limit q:

$$\lim\frac{m}{n^2} = q.$$

We have

$$\lim n\sin\frac{\pi}{n} = \pi,$$

but on the other hand, by virtue of the assumption we have made,

$$\lim m\left(1 - \cos\frac{\pi}{n}\right) = \lim m \cdot 2\sin^2\frac{\pi}{2n} = \lim\frac{\pi^2}{2}\frac{m}{n^2} = \frac{\pi^2}{2}q.$$

Hence,

$$\lim\sum_{m,n} = 2\pi R\sqrt{\left(\frac{\pi^4R^2}{4}q^2 + H^2\right)},$$

and we see that this limit depends essentially on the value of q, i.e. on the way in which m and n increase together. When $q = 0$, and only in this case, the given limit is equal to $2\pi RH$ (the value of the area derived in a school course in geometry), but together with q it can even be made equal to infinity. Thus, when the numbers m and n increase to infinity independently of each other, a limit for the area $\sum_{m,n}$ does not in fact exist, and the surface of the cylinder, for the given definition, appears to have no area.

It is important to make quite clear the way in which the situation differs in the case of a broken curve, inscribed in a curve, from the case of a many-sided surface inscribed in a curved surface. For simplicity we will assume that the curve and the curved surface which we are discussing are smooth. Then, provided that the chords forming the broken curve are sufficiently small, the direction of each of them differs by as little as we please from the direction of the tangent at any point of the corresponding arc. Therefore, such an infinitely small chord can, with ever-increasing accuracy, be taken as a replacement for the corresponding element of the arc. However, a many-sided area taken as small as we please, whose vertices lie on the curved surface, may, in its position in space, be nowhere near the tangent plane to the surface; in this case it clearly cannot replace the element of the surface. This fact is very well illustrated by the example we have just considered: the tangent planes to the cylindrical surface are all vertical, but the triangular faces of the inscribed surface for large q become almost horizontal, with a shallow slope.

365. The area of a surface given by an explicit equation. In view of the fact mentioned in the preceding section, we need to look for another way to justify the concept of the area of a curved surface. We mention one such method, which, although simple, is somewhat artificial, where we begin with the case of a surface (S) given by the explicit equation

$$z = f(x, y). \tag{1}$$

Let x, y vary in a four-sided domain (D) on the xy plane, and let z have continuous partial derivatives

$$p = \frac{\partial z}{\partial x} \quad \text{and} \quad q = \frac{\partial z}{\partial y}$$

in this domain.

We divide up the domain (D) by means of a net of curves (with area equal to zero) into the elements

$$(D_1), (D_2,), \ldots, (D_n)$$

and we consider one of these elements, (D_i). If we construct on the contour of this partial domain (as on a directrix) a cylindrical surface with generators parallel to the z-axis, then it cuts out an element (S_i) on the surface (S). Taking any point $P_i(x_i, y_i)$ within the limits of (D_i), we draw, at the corresponding point $M_i(x_i, y_i, z_i)$, where

$$z_i = f(x_i, y_i),$$

the tangent plane to the surface (S). The cylindrical surface mentioned above also cuts out an elementary figure (T_i) (Fig. 55) on this plane,

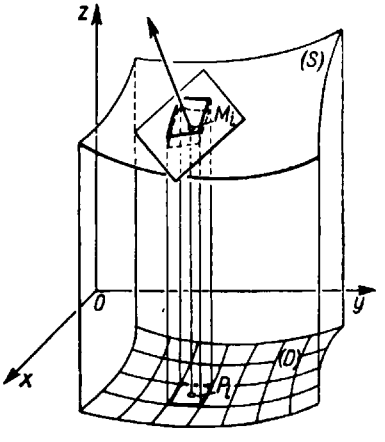

FIG. 55.

whose area T_i serves as an approximation to the area of the element (S_i). Thus the sum of all such areas

$$\sigma = \sum_{i=1}^{n} T_i$$

can be regarded as an approximation to the area of the surface (S). *We define the area S of this surface as the limit*

$$S = \lim\sigma = \lim \sum_{i=1}^{n} T_i$$

with the hypothesis that the diameters of all the elements (S_i) tend to zero or, equivalently, the diameters of all the plane elements (D_i)

Now it is easy to show that S can be expressed as a double integral

$$S = \iint\limits_{(D)} \frac{1}{|\cos\nu|}\, dx\, dy, \tag{2}$$

where, as always, ν denotes the angle made by the normal to the surface with the z-axis†.

In fact, if ν_i is the value of ν corresponding to the point M_i, then for the areas of the plane figures (T_i) and (D_i), of which the second is an orthogonal projection of the first, we will have the relation

$$D_i = T_i|\cos\nu_i|,$$

whence

$$T_i = \frac{1}{|\cos\nu_i|}\, D_i.$$

The sum

$$\sigma = \sum_{i=1}^{n} T_i = \sum_{i=1}^{n} \frac{1}{|\cos\nu_i|}\, D_i$$

is the integral sum for the double integral (2) and, hence, tends to it as limit for the given passage to limits, which is what we wished to prove.

If we recall the formulae (8), Sec. 361 (the third equation), then the result obtained can be rewritten thus:

$$S = \iint\limits_{(D)} \sqrt{(1 + p^2 + q^2)}\, dx\, dy; \tag{2a}$$

the actual existence of the integral (2) or (2a) follows from the hypothesis of the continuity of the partial derivatives p and q.

Remark. The essential defects of this simple definition of the area S are, first, its applicability only to surfaces which can be represented by explicit equations, and, secondly, its apparent de-

† We call the reader's attention to the complete analogy between this expression (2) for the area of a curved surface, given by an explicit equation (1), and the expression (7), Sec. 328 for the length of an arc of a curved line—also in the case when it is given explicitly by the equation $y = f(x)$.

pendence on the choice of the coordinate system. We shall return to this in the following section.

366. The area of a surface in the general case. Now we will consider a simple smooth surface (S), given parametrically.

Let M be any of its points, and at it, say, $C \neq 0$. Then, from what we said in Sec. 360, we can assert the following: there exists a part (s) of the surface (S), surrounding the point M and possessing the properties:

(1) the surface (s) can be represented by an explicit equation of the form (1);

(2) if we denote by (δ) the corresponding part of the domain (Δ) on the uv plane, then at (δ) the determinant $C \neq 0$.

This is true for every point M of the surface, provided only that if either of the other determinants, A or B, is distinct from 0 at this point, then the explicit equation of the form (1) is replaced, respectively, by an explicit equation of another type

$$x = h(y, z) \quad \text{or} \quad y = g(z, x).$$

Hence it follows (we will not stop to prove this) that the whole surface (S) is divided up into a finite number of pieces like (s). Now we will calculate the area of the piece (s) in accordance with the definition in the preceding section.

If (d) is the projection of (s) on the xy plane, then, as we know [see (2)], its area is

$$s = \iint\limits_{(d)} \frac{dx\,dy}{|\cos\nu|}. \tag{3}$$

Because the points of the domain (d) can be related to the points of the domain (δ) by a one-one correspondence and the other requirements of Secs. 352 and 356 are satisfied, then in the integral (3) we can change to the variables u, v, by formula (15), Sec. 356. Taking into account that the functional determinant

$$\frac{D(x, y)}{D(u, v)} = C,$$

and

$$|\cos\nu| = \frac{|C|}{\sqrt{(A^2 + B^2 + C^2)}}$$

[see Sec. 360, (7)], as a result we obtain

$$s = \iint\limits_{(\delta)} V(A^2 + B^2 + C^2)\, du\, dv. \tag{4}$$

It is remarkable that this expression is symmetric with respect to A, B, C, and nothing in it reflects the fact that we presupposed that the determinant C was distinct from zero in (δ), and the surface (s) is represented by an explicit equation of the form (1): exactly the same result is obtained under the other possible hypotheses.

Now we define the area S of the whole surface (S) as the sum of the areas s of its separate parts (s). Adding together all the equations of the form (4), we arrive at the final formula

$$S = \iint\limits_{(\Delta)} V(A^2 + B^2 + C^2)\, du\, dv, \tag{5}$$

which is clearly independent of the parts into which we divide the surface (S).

Remark. We will show that *the value* (5) *of the area does not in fact depend on the choice of the parametric representation of the surface* (S) *under consideration.*

Let us go over from the parameters u, v, whose domain of variation is (Δ), to the parameters $\overline{u}, \overline{v}$, with domain of variation $(\overline{\Delta})$, by means of the formulae

$$u = U(\overline{u}, \overline{v}), \quad v = V(\overline{u}, \overline{v}),$$

which establishes a one–one correspondence between these domains (the functions U and V are presupposed continuous together with their derivatives). Then the surface can be expressed by the new equations

$$x = \overline{x}(\overline{u}, \overline{v}), \quad y = \overline{y}(\overline{u}, \overline{v}), \quad z = \overline{z}(\overline{u}, \overline{v});$$

suppose that it has no singularity in this representation. Letting

$$J = \frac{D(u, v)}{D(\overline{u}, \overline{v})},$$

we have†

$$\overline{A} = AJ, \ \overline{B} = BJ, \ \overline{C} = CJ$$

[Sec. 325]. Hence, amongst other things, it is clear that J in $(\overline{\Delta})$ is distinct from zero, because otherwise the surface would have singularities in its new repre-

† The notation is self-evident.

sentation. Now by the formula for the change of variables we obtain at once

$$S = \iint\limits_{(\Delta)} V(A^2 + B^2 + C^2)\,|J|\,d\bar{u}\,d\bar{v} = \iint\limits_{(\bar{\Delta})} V(\bar{A}^2 + \bar{B}^2 + \bar{C}^2)\,d\bar{u}\,d\bar{v},$$

which is what we wished to prove.

As we wish to write formula (5) in another form (in which it is usually used), let us square the matrix

$$\begin{pmatrix} x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \end{pmatrix}$$

and form the determinant

$$\begin{vmatrix} x'^2_u + y'^2_u + z'^2_u & x'_u x'_v + y'_u y'_v + z'_u z'_v \\ x'_u x'_v + y'_u y'_v + z'_u z'_v & x'^2_v + y'^2_v + z'^2_v \end{vmatrix};$$

by a well-known theorem of algebra [cf. Sec. 326] it is precisely equal to $A^2 + B^2 + C^2$. We usually set

$$x'^2_u + y'^2_u + z'^2_u = E, \quad x'_u x'_v + y'_u y'_v + z'_u z'_v = F,$$

$$x'^2_v + y'^2_v + z'^2_v = G;$$

these are the so-called *Gauss coefficients* of the surface, which play an important part in differential geometry. In this notation

$$A^2 + B^2 + C^2 = EG - F^2,$$

so that formula (5) can also be written thus:

$$S = \iint\limits_{(\Delta)} V(EG - F^2)\,du\,dv. \tag{5a}$$

The expression

$$V(A^2 + B^2 + C^2)\,du\,dv = V(EG - F^2)\,du\,dv \tag{6}$$

is called *the element of area in curvilinear coordinates.*

Remark. With the help of formula (5a), by using elementary theory from the field of vector analysis, it is easy to show that *the expression for the area S does not depend on the choice of the coordinate system,* because the expressions E, F, G keep the same values under a transformation of coordinates.

We have limited ourselves until now to the case of a simple and smooth surface. If a surface does not fall into this category, but can be divided up into a finite number of simple and smooth curves,

then we will call its area the sum of the areas of the separate pieces. Here it is easy to show that the area so defined does not in fact depend on the way in which the given surface is divided up into pieces of the required type. If the whole of the given surface is characterized by parametric equations, although the one-one correspondence is violated at separate points or along separate lines, then its area in the general case is expressed as before by formula (5) or (5a) [cf. Sec. 354, *(4)*].

A surface which has an area is called *quadrable.*

367. Examples. *(1)* To find the area of the part of the spherical surface $x^2+y^2+z^2=R^2$ cut out from it by the cylinder $x^2+y^2=Rx$ (the upper and lower bases of the "Viviani solid", see Sec. 358, *(3)*; Fig. 46).

Solution. For the upper base we have

$$z=\sqrt{(R^2-x^2-y^2)}, \quad p=-x/z, \quad q=-y/z,$$

$$\sqrt{(1+p^2+q^2)}=\frac{R}{\sqrt{(R^2-x^2-y^2)}}$$

and, hence,

$$S=2R\iint\limits_{(D)}\frac{dx\,dy}{\sqrt{(R^2-x^2-y^2)}},$$

where the domain of integration is the circle bounded by the circumference $x^2+y^2=Rx$.

Changing to polar coordinates, we obtain [cf. Sec. 358, *(3)*]

$$S=4R\int_0^{\pi/2}d\theta\int_0^{R\cos\theta}\frac{r\,dr}{\sqrt{(R^2-r^2)}}.$$

Carrying out the integration, we finally find that $S=4R^2\{(\pi/2)-1\}$.

Because the area of the surface of a hemisphere is equal to $2\pi R^2$, the area of the surface of the part of the hemisphere which remains after the "Viviani solid" has been excluded will be equal to $4R^2$ and, hence, is expressed in terms of the radius R without involving any irrationals. Viviani connected this fact, which had attracted his attention, with a problem suggested by his contemporaries. We note that the volume of the given part of the hemisphere can also be expressed without involving irrationals: it is equal to $8R^3/9$ [Sec. 358, *(3)*].

Remark. The actual carrying out of the calculation requires some reservations in connection with the fact that at the point $(R, 0, 0)$ the derivatives p and q, and with them the expressions under the integral, suffer discontinuities; see *(2)*.

(2) To solve problem *(1)*, using the parametric representation of the spherical surface:

$$x = R\sin\varphi\cos\theta, \quad y = R\sin\varphi\sin\theta, \quad z = R\cos\varphi$$
$$(0 \leqslant \varphi \leqslant \pi, \quad 0 \leqslant \theta \leqslant 2\pi).$$

In the matrix of the derivatives

$$\begin{pmatrix} R\cos\varphi\cos\theta & R\cos\varphi\sin\theta & -R\sin\varphi \\ -R\sin\varphi\sin\theta & R\sin\varphi\cos\theta & 0 \end{pmatrix}$$

it is easy to find the Gauss coefficients of the sphere:

$$E = R^2, \quad F = 0, \quad G = R^2\sin^2\varphi,$$

so that $\sqrt{(EG - F^2)} = R^2\sin\varphi$.

We will limit ourselves to the study of a quarter of the surface under examination, that lying in the first octant. For the points of the "Viviani curve", i.e. the curve of intersection of the sphere and the cylinder (within the limits of the first octant), $\varphi + \theta = \pi/2$.

In fact, substituting the expressions for x and y in terms of φ and θ in the equation of the cylinder $x^2 + y^2 = Rx$, we obtain $\sin\varphi = \cos\theta$, and because for the points under consideration, clearly, $0 \leqslant \theta \leqslant \pi/2$, $0 \leqslant \varphi \leqslant \pi/2$, it follows from this that $\varphi + \theta = \pi/2$.

By fixing, from what has been said, the limits of variation of the parameters φ and θ, we obtain, by formula (5a)

$$S = 4R^2 \int_0^{\pi/2} d\theta \int_0^{(\pi/2)-\theta} \sin\varphi\, d\varphi = 4R^2\left(\frac{\pi}{2} - 1\right).$$

We arrive at the same result, but this time we avoid the discontinuities o the function under the integral.

(3) To find an expression for *the area of the surface of rotation*

$$x = \varphi(u)\cos v, \quad y = \varphi(u)\sin v, \quad z = \psi(u)$$
$$(\alpha \leqslant u \leqslant \beta, \quad 0 \leqslant v < 2\pi)$$

[Sec. 360, *(2)*]. From the matrix of partial derivatives

$$\begin{pmatrix} \varphi'(u)\cos v, & \varphi'(u)\sin v, & \psi'(u) \\ -\varphi(u)\sin v, & \varphi(u)\cos v, & 0 \end{pmatrix}$$

we construct the expression

$$\sqrt{(EG - F^2)} = \varphi(u)\ \sqrt{\{[\varphi'(u)]^2 + [\psi'(u)]^2\}},$$

so that the area of the surface is represented by the formula

$$S = 2\pi \int_\alpha^\beta \varphi(u)\sqrt{\{[\varphi'(u)]^2 + [\psi'(u)]^2\}}\, du.$$

[Cf. formula (6a), Sec. 205.]

§ 3. Surface integrals of the first type

368. The definition of a surface integral of the first type. Surface integrals of the first type are the same sort of natural generalization of double integrals as curvilinear integrals of the first type are with respect to simple definite integrals.

We construct this generalization as follows. At the points of some two-sided smooth (or piece-wise smooth) surface (S), bounded by a piece-wise smooth contour, let there be defined a function $f(M) = f(x, y, z)$. We divide up the surface (S) by means of a net of arbitrarily drawn piece-wise smooth curves into the parts $(S_1), (S_2), \ldots, (S_n)$. Taking an arbitrary point $M_i(x_i, y_i, z_i)$ in each part (S_i) $(i = 1, 2, \ldots, n)$, we calculate the value of the function

$$f(M_i) = f(x_i, y_i, z_i)$$

at this point, and multiplying it by the area S_i of the corresponding part of the surface, we form the sum of all such products:

$$\sigma = \sum_{i=1}^{n} f(M_i) S_i = \sum_{i=1}^{n} f(x_i, y_i, z_i) S_i,$$

which we will call an integral sum—like the many other sums considered earlier.

The finite limit of this sum as the diameters of all the parts (S_i) tend to zero is called the surface integral (of the first type†) of the function $f(M) = f(x, y, z)$ over the surface (S) and is denoted by the symbol

$$\iint_{(S)} f(M)\,dS = \iint_{(S)} f(x, y, z)\,dS, \tag{1}$$

where dS represents the elementary areas S_i.

369. The reduction to an ordinary double integral. We will limit ourselves to the case of a simple and smooth surface (S).

Whatever the function $f(x, y, z)$ may be, which is continuous at the points of the surface (S), the integral (1) exists, and the equation

$$\iint_{(S)} f(x, y, z)\,dS = (\mathrm{R}) \iint_{(\Delta)} f(x(u, v), y(u, v), z(u, v)) \sqrt{(EG - F^2)}\,du\,dv \tag{2}$$

holds.

† To distinguish it from surface integrals of the second type considered below.

Thus, *to reduce the surface integral of the first type to an ordinary double integral we need only replace the coordinates x, y, z by their expressions in terms of the parameters, and the element of area dS by its expression in curvilinear coordinates.*

We turn to the proof of the assertion which we have just stated.

As has already been noted, the subdivision of the surface (S) by piece-wise smooth curves corresponds to a similar subdivision of the domain (Δ), and conversely. Also, if the diameters of the parts of (S) tend to zero, this is also true with respect to the diameters of the parts of (Δ), and conversely.

In a corresponding way we divide up the surface (S) into the parts $(S_1), (S_2), \ldots, (S_n)$, and the domain (Δ) into the parts (Δ_1), $(\Delta_2), \ldots, (\Delta_n)$ and we choose in each part (S_i) a point (x_i, y_i, z_i), and in the part (Δ_i) a point (u_i, v_i), which correspond to one another, so that

$$x_i = x(u_i, v_i), \quad y_i = y(u_i, v_i), \quad z_i = z(u_i, v_i). \tag{3}$$

Now we form the integral sum for the integral (1):

$$\sigma = \sum_{i=1}^{n} f(x_i, y_i, z_i) S_i.$$

By the general formula (5a), Sec. 366

$$S_i = \iint\limits_{(\Delta_i)} \sqrt{(EG - F^2)}\, du\, dv$$

holds. Applying the mean value theorem, we obtain

$$S_i = \left[\sqrt{(EG - F^2)}\right]_{\substack{u = \bar{u}_i \\ v = \bar{v}_i}} \Delta_i,$$

where $(\bar{u}_i, \bar{v}_i)$ is some point in the domain (Δ_i).

With the help of this expression for (S_i) and using (3), we can rewrite the sum σ thus:

$$\sigma = \sum_{i=1}^{n} f(x(u_i, v_i),\ y(u_i, v_i),\ z(u_i, v_i)) \left[\sqrt{(EG - F^2)}\right]_{\substack{u = \bar{u}_i \\ v = \bar{v}_i}} \Delta_i.$$

In this form it recalls the integral sum for the second of the integrals (2):

$$\sigma^* = \sum f(x(u_i, v_i),\ y(u_i, v_i),\ z(u_i, v_i)) \left[\sqrt{(EG - F^2)}\right]_{\substack{u = u_i \\ v = v_i}} \Delta_i.$$

The difference between the sums σ and σ^* lies in the fact that in the latter both the compound function $f(\ldots)$ and the root $\sqrt{(\ldots)}$ are evaluated every time for the same arbitrarily chosen point (u_i, v_i), but in the former the function $f(\ldots)$ is taken at the point (u_i, v_i), and the expression $\sqrt{(\ldots)}$ at the point $(\bar{u}_i, \bar{v}_i)$ (which is fixed by the mean value theorem and is not arbitrary).

We will consider the difference between the two sums:

$$\sigma - \sigma^* = \sum_i f(\ldots)\{[\sqrt{(EG - F^2)}]_{\substack{u=\bar{u}_i \\ v=\bar{v}_i}} - [\sqrt{(EG - F^2)}]_{\substack{u=u_i \\ v=v_i}}\} \Delta_i.$$

Let $\varepsilon > 0$ be an arbitrarily small number. By virtue of the (uniform) continuity of the function $\sqrt{(EG - F^2)}$, for sufficiently small diameters of the domains (Δ_i) we will have

$$|[\sqrt{(EG - F^2)}]_{\substack{u=\bar{u}_i \\ v=\bar{v}_i}} - [\sqrt{(EG - F^2)}]_{\substack{u=u_i \\ v=v_i}}| < \varepsilon.$$

Taking into account the boundedness of the continuous function f:

$$|f(x, y, z)| \leqslant L,$$

it is easy to arrive at the approximation

$$|\sigma - \sigma^*| < \varepsilon L \Delta,$$

so that

$$\lim(\sigma - \sigma^*) = 0.$$

Hence it is clear that from the existence of a limit for the first of these sums follows the existence of a limit, equal to it, for the other. This proves our assertion.

If the surface (S) is given by an explicit equation:

$$z = z(x, y),$$

then formula (2) takes the form

$$\iint_{(S)} f(x, y, z)\, dS = (\mathrm{R}) \iint_{(D)} f(x, y, z(x, y))\, \sqrt{(1 + p^2 + q^2)}\, dx\, dy, \qquad (4)$$

where (D) denotes the projection of the surface (S) on the xy plane.

Because

$$\sqrt{(1 + p^2 + q^2)} = \frac{1}{|\cos \nu|}$$

(where ν, as usual, is the angle between the normal to the surface and the z-axis), then formula (4) can be rewritten thus:

$$\iint\limits_{(S)} f(x, y, z)\,dS = (\mathrm{R}) \iint\limits_{(D)} f(x, y, z(x, y)) \frac{dx\,dy}{|\cos \nu|}. \tag{5}$$

We have presupposed up to now that the surface (S), over which the integrals extend, is simple and smooth. Our results carry over easily to the general case of a surface consisting of a finite number of such pieces.

370. Mechanical applications of surface integrals of the first type. *(1).* With the help of the integrals mentioned we can determine the values of the mass, moments, coordinates of the centre of gravity, and so on, for surfaces covered with a distribution of matter, along which are distributed masses with density defined at each point of the surface.

In fact there is nothing new here as compared with the case of a plane distribution of matter, considered above.

(2) The attraction of a simple layer. Surface integrals of the first type aturally lead us to consider the study of the attraction of masses distributed over a surface.

On the surface (S) let there be continuously distributed masses with given surface density $\varrho(M) = \varrho(x, y, z)$ at each point $M(x, y, z)$ of the surface†. Further, at the point $A(\xi, \eta, \zeta)$ (outside the surface) let there be situated a unit mass. It is required to determine with what magnitude and direction of force $\vec{F}$ the point A is attracted towards the surface (S), if Newton's law of attraction is taken as our basis (the law of universal gravitation).

If the point A were attracted by just one particle, mass m, situated at $M(x, y, z)$, then the magnitude of the force of attraction would be equal to

$$F = \frac{m}{r^2}‡,$$

where r is the distance $\overline{AM}$, i.e.

$$r = \sqrt{[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]}.$$

Because this force is directed from A to M, its direction cosines will be

$$\frac{x-\xi}{r}, \quad \frac{y-\eta}{r}, \quad \frac{z-\zeta}{r}$$

† In this case we speak of a simple layer (as distinct from a double layer, which we will not consider).

‡ As usual, the "constant of attraction", i.e. the factor of proportionality in Newton's formula (which depends on the choice of units), we replace by unity, to simplify our working.

and, hence, the projection of the force of attraction $\vec{F}$ on the axes of coordinates can be expressed thus:

$$F_x = m\frac{x-\xi}{r^3}, \quad F_y = m\frac{y-\eta}{r^3}, \quad F_z = m\frac{z-\zeta}{r^3}. \tag{7}$$

In the case of a system of attracting particles these expressions would be replaced by the sums of similar expressions; finally, when there is a continuous distribution of masses over a surface we have integrals instead of the sums.

Applying our usual method of reasoning, we can consider an element dS of the surface with mass $\varrho\, dS$, as if it were situated at one of its points $M(x, y, z)$. The attraction towards it at the point A will have the projections on the axes [cf. (7)]:

$$dF_x = \varrho\frac{x-\xi}{r^3}\,dS, \quad dF_y = \varrho\frac{y-\eta}{r^3}\,dS, \quad dF_z = \varrho\frac{z-\zeta}{r^3}\,dS,$$

where r denotes the distance $\overline{AM}$, expressed by formula (6). Now it only remain to "sum" these expressions, which leads to the following formula for the projection of the force $\vec{F}$ of the attraction of a simple layer on the axes:

$$F_x = \iint\limits_{(S)} \varrho\frac{x-\xi}{r^3}\,dS, \quad F_y = \iint\limits_{(S)} \varrho\frac{y-\eta}{r^3}\,dS, \quad F_z = \iint\limits_{(S)} \varrho\frac{z-\zeta}{r^3}\,dS. \tag{8}$$

These determine the force $\vec{F}$ completely both in magnitude and direction.

If the point A, which is attracted, were itself to lie on the surface (S), then the projections of the attraction on the axes would be expressed as before by the integrals (8), but in this case these integrals would be improper, in so far as near the point A the functions under the integrals all cease to be bounded.

(3) The potential of a simple layer. In the case of one attracting point $M(x, y, z)$, as we saw, the projections of the force of attraction are expressed by (7). It is easy to see that these projections are the partial derivatives with respect to ξ, η and ζ of the function

$$W(\xi, \eta, \zeta) = m/r,$$

which is called the *Newtonian potential* at the point A of the field of the point M. [Cf. Sec. 351, *(1)*.]

In the case of a field formed by a system of particles, the potential would be expressed by a sum of fractions of this type, where the derivatives of the potential with respect to ξ, η, ζ give, as before, the projections of the force of attraction on the axes.

Hence, it is natural to go on to such an expression for the *potential of a simple layer*, distributed over the surface (S), with density ϱ, at the point A:

$$W(\xi, \eta, \zeta) = \iint\limits_{(S)} \varrho\frac{dS}{r}. \tag{9}$$

There only arises the question of whether the fundamental properties

$$\frac{\partial W}{\partial \xi} = F_x, \quad \frac{\partial W}{\partial \eta} = F_y, \quad \frac{\partial W}{\partial \zeta} = F_z \tag{10}$$

are preserved for this potential, where F_x, F_y, F_z are the projections of the force of attraction $\vec{F}$ of the simple layer on the axes and are defined by the formulae (8).

If the point A does not lie on the surface, so that there is no violation of the continuity, then it is easy to show that Leibnitz's rule is applicable to the integral (9) in its differentiation with respect to ξ, η or ζ (for this it is only necessary to repeat the reasoning which we already know). In this way we justify the relation (10) for the case of the distribution of matter under consideration.

Examples. {*1*} To find the force of attraction from a point to a homogeneous ($\varrho = \text{const}$) spherical layer.

Solution. Let the centre of the sphere lie at the origin of coordinates, and let the point A (of mass 1) which is being attracted be situated on the positive z-axis at a distance a from the centre. The projections F_x and F_y of the force of attraction on the x and y axes, clearly, are equal to zero. Further, we have

$$F_z = \iint\limits_{(S)} \varrho \frac{z-a}{r^3}\, dS$$

(r is the distance between the point A and an arbitrary point M on the sphere). If we change to spherical coordinates:

$$x = R\sin\varphi\cos\theta, \quad y = R\sin\varphi\sin\theta, \quad z = R\cos\varphi,$$

then

$$dS = R^2\sin\varphi\, d\varphi\, d\theta, \quad r = \sqrt{(R^2 + a^2 - 2Ra\cos\varphi)}$$

and

$$F_z = 2\pi R^2\varrho \int\limits_0^{\pi} \frac{(R\cos\varphi - a)\sin\varphi\, d\varphi}{(R^2 + a^2 - 2Ra\cos\varphi)^{3/2}}.$$

By the substitution $R^2 + a^2 - 2Ra\cos\varphi = t^2$ we transform this expression to

$$F_z = \frac{\pi R}{a^2}\varrho \int\limits_{|R-a|}^{R+a} \left(\frac{R^2 - a^2}{t^2} - 1\right) dt = -\frac{\pi R}{a^2}\varrho\left(2R - \frac{R^2 - a^2}{|R-a|} - |R-a|\right).$$

Now we consider two positions.

(1) Let $a < R$; in this case $|R-a| = R-a$, the expression in the brackets becomes zero and

$$F_z = 0.$$

Thus, *a point situated inside a homogeneous spherical layer does not experience any attraction from the latter.*

(2) If $a > R$, then $|R-a| = -(R-a)$, so that

$$F_z = -\frac{4\pi R^2\varrho}{a^2}.$$

Therefore, *a point situated outside a homogeneous spherical layer experiences the same attraction from the latter as it would if the whole mass* $m = 4\pi R^2\varrho = S\varrho$ *of the layer were concentrated at the centre.*

{2} To find the potential of a homogeneous spherical layer at an arbitrarily chosen point.

Solution. With our previous notation we have

$$W(a) = \iint_{(S)} \varrho \frac{dS}{r} = 2\pi R^2\varrho \int_0^{\pi} \frac{\sin\varphi\, d\varphi}{\sqrt{(R^2 + a^2 - 2Ra\cos\varphi)}}$$

$$= \frac{2\pi R}{a}\varrho \int_{|R-a|}^{R+a} dt = \frac{2\pi R}{a}\varrho(R + a - |R - a|).$$

If $a < R$, then

$$W(a) = 4\pi R\varrho,$$

so that *inside a homogeneous spherical layer its potential is constant.*

However, when $a > R$ we will have

$$W(a) = \frac{4\pi R^2\rho}{a},$$

i.e. *the potential created by the spherical layer in outside space is unaltered if the whole mass is concentrated at the centre.*

Remark. In the case $a = R$ in both problems we would have to deal with a so-called improper surface integral, in so far as the function under the integral becomes infinite.

§ 4. Surface integrals of the second type

371. The definition of surface integrals of the second type. This new integral form is constructed like the curvilinear integral of the second type.

In that case we started from a directed (oriented) curve and, dividing it up into elements, each of these elements being correspondingly directed, we projected them onto the coordinate axis. The projection obtained was also directed, and we took its length with a plus or minus sign depending on whether its direction coincided with the direction along the axis or not.

In an analogous way we now consider the two-sided surface (S), which is smooth or piece-wise smooth, and we decide on one of its two sides; as we have seen [Sec. 362], *this is equivalent to the choice of a definite orientation on the surface.*

In order to be specific we presuppose from the start that the surface is given by the explicit equation

$$z = z(x, y),$$

where the point (x, y) varies in a domain (D) in the xy plane, bounded by a piece-wise smooth contour. Then it is possible for us to choose between the upper and lower sides of the surface†. In the first case a closed curve on the surface has a counterclockwise direction, if we look at it from above, and in the second case it has the opposite direction.

If the surface is divided up into elements and each of these elements, which is correspondingly oriented, is projected on the xy plane, then the direction of the circuit of the contour of the figure which is projected determines the direction of the circuit of the contour of the projection. This direction will coincide with a counterclockwise rotation, i.e. it will correspond to the orientation of the xy plane itself, if we decide on the upper side of the surface (S); in this case we will take the area of the projection with a plus sign. In the case of the lower side the rotation will be in the opposite direction, and the area of the projection will be taken with the minus sign [cf. Sec. 357].

Now at the points of the given surface (S) let there be defined some function $f(M) = f(x, y, z)$. Dividing up the surface by a net of piece-wise smooth curves into the elements

$$(S_1),\ (S_2),\ \ldots,\ (S_n),$$

we choose in each element (S_i) a point $M_i(x_i, y_i, z_i)$. Then we calculate the value of the function $f(M_i) = f(x_i, y_i, z_i)$ and multiply it by the area D_i of the projection of the element (S_i) on the xy plane, choosing the sign by the rule mentioned above. Finally, we form the sum (also, of its kind, an integral sum)

$$\sigma = \sum_{i=1}^{n} f(M_i)\, D_i = \sum_{i=1}^{n} f(x_i, y_i, z_i)\, D_i. \tag{1}$$

The finite limit of this sum as the diameters of all the parts (S_i) tend to zero is called the surface integral (of the second type) of

$$f(M)\, dx\, dy = f(x, y, z)\, dx\, dy,$$

† See footnote on p. 320.

extended over a chosen side of the surface (S), and it is denoted by the symbol

$$I = \iint\limits_{(S)} f(M)\,dx\,dy = \iint\limits_{(S)} f(x, y, z)\,dx\,dy \tag{2}$$

(here $dx\,dy$ represents the area of the projection of the element of the surface on the xy plane).

However, this symbol contains no precise indication of which side of the surface we actually have in view, so that each time we have to give a special indication of the side we have in mind. It follows from the definition itself that if the side of the surface under consideration is changed to the opposite side, the integral changes sign.

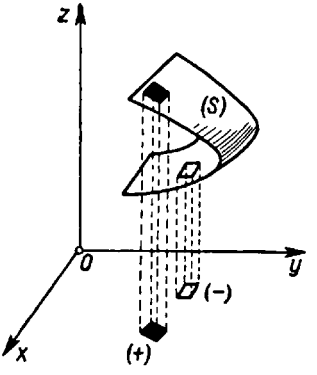

FIG. 56.

If the surface does not have the special form indicated, then from now on we will always assume that it consists of a finite number of pieces, bounded by piece-wise smooth contours and either having this form or representing part of a cylindrical surface with generators parallel to the z-axis (whose directrix has zero area on the xy plane). In the case of an element lying on a piece of the first type we already know how to write the sign of the area of its projection; these signs may be different for different elements, if one of the elements of the surface lies above and the other below (Fig. 56). As for an element lying on the cylindrical surface which we mentioned, its projection shrinks to a line, and it is not possible to speak of its

sign†. In what remains the definition of a surface integral is constructed for the general case in the same way as above.

By altering the roles of the axes (corresponding to which our requirements for the surface also change), we could project the elements of the surface on the yz or zx plane instead of on the xy plane. The other two surface integrals of the second type obtained in this way are:

$$\iint\limits_{(S)} f(x, y, z)\, dy\, dz \quad \text{or} \quad \iint\limits_{(S)} f(x, y, z)\, dz\, dx. \tag{2*}$$

In applications we usually meet a combination of the integrals of all these forms:

$$\iint\limits_{(S)} P\, dy\, dz + Q\, dz\, dx + R\, dx\, dy,$$

where P, Q, R are functions of (x, y, z) defined at the points of the surface (S). Once again we emphasize that *in all cases the surface (S) is presumed to be two-sided and that the integrals extend over a definite side of it.*

372. The reduction to an ordinary double integral. We will presuppose that the function f is continuous at the points of the surface (S).

(1) We will consider first the fundamental case, when the surface (S) is given by the explicit equation

$$z = z(x, y) \qquad ((x, y) \text{ from } (D)),$$

where the function z is continuous together with its partial derivatives

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}.$$

If the integral (2) is taken over the upper side of the surface, then in the integral sum (1) all the D_i are positive. Substituting in this sum instead of z_i its value $z(x_i, y_i)$, we reduce it to the form

$$\sigma = \sum_{i=1}^{n} f(x_i, y_i, z(x_i, y_i))\, D_i,$$

† We will avoid "irregular" elements, belonging to different pieces, from which the surface is constructed (we can simply neglect them).

which it is easy to recognize as the integral sum for the ordinary double integral

$$(R)\iint\limits_{(D)} f(x, y, z(x, y))\,dx\,dy.$$

By taking limits, we establish the existence of the integral (2) and the equation

$$\iint\limits_{(S)} f(x, y, z)\,dx\,dy = (R)\iint\limits_{(D)} f(x, y, z(x, y))\,dx\,dy. \tag{3}$$

If the integral extends over the lower side of the surface (S), then we will have, clearly,

$$\iint\limits_{(S)} f(x, y, z)\,dx\,dy = -(R)\iint\limits_{(D)} f(x, y, z(x, y))\,dx\,dy. \tag{3a}$$

Now it is easy to establish (for the case under consideration) *the connection between the surface integrals of both types.* First let us discuss (for an integral of the second type) the upper side of the surface. If, assuming that the angle ν is acute, we replace the function $f(x, y, z)$ by $f(x, y, z)\cos\nu$ in formula (5), Sec. 369, then we can write:

$$(R)\iint\limits_{(D)} f(x, y, z(x, y))\,dx\,dy = \iint\limits_{(S)} f(x, y, z)\cos\nu\,dS.$$

Hence, by (3), we obtain the desired formula:

$$\iint\limits_{(S)} f(x, y, z)\,dx\,dy = \iint\limits_{(S)} f(x, y, z)\cos\nu\,dS. \tag{4}$$

When replacing the upper by the lower side of the surface, at the same time we change the sign on the left side of equation (4). If at the same time as this we understand ν to be this time an obtuse angle made by the normal with the z-axis, directed downwards, then the cosine, and with it the integral, will also change sign, so that the equation holds.

(2) If (S) is part of a cylindrical surface with generators parallel to the z-axis, then all its elements have zero projection, so that in this case

$$\iint\limits_{(S)} f(x, y, z)\,dx\,dy = 0. \tag{5}$$

Clearly formula (4) also holds here: because $\cos\nu = 0$, the right side of this formula will be zero.

(3) Finally, if the surface (S) consists of a finite number of pieces, considered in *(1)* or *(2)*, then, adding together the formulae of the form (4) corresponding to the separate pieces, we can ascertain that the formula (4) is also valid in the general case.

(4). Analogous formulae can be obtained for the surface integrals (2*). By adding together all three formulae written down, respectively, for the arbitrary continuous functions P, Q, R, we arrive at a general formula, connecting the surface integrals of the second and first types:

$$\iint\limits_{(S)} P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy = \iint\limits_{(S)} (P\cos\lambda + Q\cos\mu + R\cos\nu)\,dS. \quad (6)$$

We emphasize that *on the right side of this appear the direction cosines of the normal corresponding to the side of the surface over which the integral on the left is taken.*

(5) If the surface (S) is given parametrically, then we can reduce the integral on the right side of formula (6)—and with it the integral on the left side—to an ordinary double integral, extended over the domain (Δ) of variation of the parameters. We presuppose first that the surface is simple and smooth; let it be bounded by the piece-wise smooth contour (L).

We choose a definite side of the surface and in so doing we establish a definite orientation on it. If the positive circuit of the contour (L) corresponds to the positive circuit of the contour (Δ) of the domain (Δ), then, as we know [Sec. 362], the direction cosines of the normal characterized by the chosen side of the surface are defined by the formulae

$$\cos\lambda = \frac{A}{+\sqrt{(A^2+B^2+C^2)}}, \quad \cos\mu = \frac{B}{+\sqrt{(A^2+B^2+C^2)}},$$

$$\cos\nu = \frac{C}{+\sqrt{(A^2+B^2+C^2)}}$$

with the sign $(+)$ before the radical. On the other hand, when we

change to the double integral with respect to the parameters u, v, the element of area dS must be replaced by the expression

$$+\sqrt{(A^2+B^2+C^2)}\,du\,dv$$

[Sec. 369]. Finally, we obtain the equation

$$\iint_{(S)} P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy = \iint_{(\Delta)} (AP + BQ + CR)\,du\,dv. \qquad (7)$$

On the right side we mean that in the functions P, Q, R we have substituted for x, y, z their expressions in terms of u and v.

Remark. This result can also be extended to the more general case of a surface consisting of simple and smooth surfaces which adjoin one another. But before the integral on the right there may appear both a plus sign and a minus sign [see Sec. 361, Remark].

373. Stokes's formula. Now we turn to the problem of finding a formula connecting a surface integral with a curvilinear integral and serving as a generalization of Green's formula; we previously considered this in Sec. 346.

At first we retain the same hypotheses with respect to the surface (S) as we had in *(5)* of the preceding section.

In some space domain containing in its interior the surface (S) let there be given a function $P(x, y, z)$, which is continuous in this domain together with its partial derivatives. Then the formula

$$\int_{(L)} P\,dx = \iint_{(S)} \frac{\partial P}{\partial z}\,dz\,dx - \frac{\partial P}{\partial y}\,dx\,dy \qquad (8)$$

holds, where the direction of the circuit of the contour (L) corresponds to the side of the surface (S) over which the integral on the right is extended.

First of all we transform the curvilinear integral along the curve (L), replacing it by the integral along the curve (Δ):

$$\int_{(L)} P\,dx = \int_{(\Delta)} P\left(\frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv\right). \qquad (9)$$

This equation is easy to verify if we take the parametric representation of the curve (Δ):

$$u = u(t), \quad v = v(t) \qquad (\alpha \leqslant t \leqslant \beta),$$

and the representation of the curve (L) in terms of it:

$$x = x(u(t), v(t)), \quad y = y(u(t), v(t)), \quad z = z(u(t), v(t)).$$

Then both integrals will reduce to the same ordinary integral with respect to the parameter

$$\int_{\alpha}^{\beta} P\left(\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}\right) dt.$$

Now we apply Green's formula to the integral on the right of (9):

$$\int_{(\Delta)} P\left(\frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv\right) = \iint_{(\Delta)} \left\{\frac{\partial}{\partial u}\left(P\frac{\partial x}{\partial v}\right) - \frac{\partial}{\partial v}\left(P\frac{\partial x}{\partial u}\right)\right\} du\,dv.$$

Because the last expression under the integral in expanded form gives

$$\left(\frac{\partial P}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial u}\right)\frac{\partial x}{\partial v} + P\frac{\partial^2 x}{\partial v\,\partial u}$$
$$-\left(\frac{\partial P}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial v}\right)\frac{\partial x}{\partial u} - P\frac{\partial^2 x}{\partial u\,\partial v}$$
$$= \frac{\partial P}{\partial z}\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right) - \frac{\partial P}{\partial y}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right),$$

then we arrive at the double integral

$$\iint_{(\Delta)} \left\{\frac{\partial P}{\partial z}B - \frac{\partial P}{\partial y}C\right\} du\,dv.$$

By formula (7) it is easy to transform this to the surface integral

$$\iint_{(S)} \frac{\partial P}{\partial z}\,dz\,dx - \frac{\partial P}{\partial y}\,dx\,dy,$$

taken in fact over the chosen side of the surface. This completes the proof of equation (8)†.

We have established this formula for a simple and smooth surface. But it is easy to extend it also to the general case of a piece-wise

† It is necessary to note that in our derivation of the formula we use the existence and continuity of the derivatives $\partial P/\partial x$, $\partial^2 x/\partial u\,\partial v$, $\partial^2 x/\partial v\,\partial u$, which do not appear in the final result. In fact, the formula holds without these hypotheses.

smooth surface, bounded by a piece-wise smooth contour: it is only necessary to write it down for each simple and smooth piece separately and to add together the equations obtained term by term.

By means of a cyclic permutation of the terms x, y, z we obtain two further analogous equations:

$$\begin{aligned} \int_{(L)} Q\,dy &= \iint_{(S)} \frac{\partial Q}{\partial x}\,dx\,dy - \frac{\partial Q}{\partial z}\,dy\,dz, \\ \int_{(L)} R\,dz &= \iint_{(S)} \frac{\partial R}{\partial y}\,dy\,dz - \frac{\partial R}{\partial x}\,dz\,dx, \end{aligned} \tag{8a}$$

where Q and R are new functions of x, y, z, satisfying the same conditions as P.

Adding together all three of the equations (8) and (8a), we obtain the desired result in its most general form:

$$\begin{aligned} &\int_{(L)} P\,dx + Q\,dy + R\,dz \\ &= \iint_{(S)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\,dx. \end{aligned} \tag{10}$$

This equation is called *Stokes's formula. Once again we emphasize that the side of the surface and the direction of the circuit of the contour depend on each other by the rule established in Sec.* 362.

If as a piece of the surface (S) we take a plane domain (D) on the xy plane, so that $z = 0$, then we obtain the formula

$$\int_{(L)} P\,dx + Q\,dy = \iint_{(D)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy,$$

which the reader will recognize as Green's formula; thus, the latter is a special case of Stokes's formula†.

† To make it simpler to recall Stokes's formula we will mention that the first summand in the integral on the right is the same as that in Green's formula, and the remaining ones are obtained from it by a cyclic permutation of the terms x, y, z and P, Q, R.

We note, finally, that the surface integral of the second type in Stokes's formula can be replaced by a surface integral of the first type. Then this formula takes the form

$$\int\limits_{(L)} P\,dx + Q\,dy + R\,dz$$
$$= \iint\limits_{(S)} \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\cos\lambda + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\cos\mu + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cos\nu\right] dS, \tag{11}$$

where $\cos\lambda$, $\cos\mu$, $\cos\nu$ denote the direction cosines of the normal, *corresponding to the chosen side of the surface.*

As an example we "check" Stokes's formula by setting

$$P = y^2 + z^2, \quad Q = z^2 + x^2, \quad R = x^2 + y^2$$

and taking as (S) the surface cut out by the cylinder $x^2 + y^2 = 2rx$ on the sphere $x^2 + y^2 + z^2 = 2Rx$ $(R > r,\ z > 0)$.

Using the parametric representation of the curve

$$x = r(1 + \cos t), \quad y = r\sin t, \quad z = \sqrt{[2r(R-r)]}\sqrt{(1+\cos t)} \qquad (0 \leqslant t \leqslant 2\pi)\dagger$$

we find for the curvilinear integral a complicated expression in the form of an ordinary integral:

$$\int_0^{2\pi} \Big\{ [r^2\sin^2 t + 2r(R-r)(1+\cos t)](-r\sin t)$$
$$+ [2r(R-r)(1+\cos t) + r^2(1+\cos t)^2]r\cos t$$
$$+ [r^2(1+\cos t)^2 + r^2\sin^2 t]\frac{1}{2}\sqrt{\left[\frac{2r(R-r)}{1+\cos t}\right]}(-\sin t)\Big\} dt.$$

But the first and third summands in the curly brackets, multiplied by dt, have the form $f(\cos t)\,d\cos t$, and their integrals, in view of the periodic nature of the cosine function, are equal to zero. Carrying out the remaining calculation, we obtain $2\pi Rr^2$.

The surface integral of the second type

$$2\iint\limits_{(S)} (y-z)\,dy\,dz + (z-x)\,dz\,dx + (x-y)\,dx\,dy,$$

† If we set $x - r = r\cos t$, $y = r\sin t$, then the geometrical meaning of the parameter t is clear; substituting these expressions in the equation of the sphere, we find that z is dependent on t. As t varies from 0 to 2π the curve is described counterclockwise, if we look at it from above.

extended over the upper side of the given surface, we transform to an integral of the first type:

$$2\iint\limits_{(S)} [(y-z)\cos\lambda+(z-x)\cos\mu+(x-y)\cos\nu]dS.$$

Because

$$\cos\lambda=\frac{x-R}{R},\quad \cos\mu=\frac{y}{R},\quad \cos\nu=\frac{z}{R},$$

then, substituting these expressions, we simplify and reduce the desired integral to the following:

$$2\iint\limits_{(S)} (z-y)dS.$$

By virtue of the symmetry of the surface with respect to the xz plane, the integral $\iint\limits_{(S)} y\,dS$ is zero. We again transform the remaining integral to an integral of the second type:

$$2\iint\limits_{(S)} z\,dS=2\iint\limits_{(S)}\frac{z}{\cos\nu}\,dx\,dy=2R\iint\limits_{(S)} dx\,dy=2\pi Rr^2.$$

374. The application of Stokes's formula to the investigation of curvilinear integrals in space. In the open domain (T) let there be given functions P, Q, R, which are continuous together with their derivatives

$$\frac{\partial P}{\partial y},\quad \frac{\partial P}{\partial z};\quad \frac{\partial Q}{\partial z},\quad \frac{\partial Q}{\partial x};\quad \frac{\partial R}{\partial x},\quad \frac{\partial R}{\partial y}.$$

With the help of Stokes's formula it is easy to establish *the conditions which are necessary and sufficient for the vanishing of the integral*

$$\int\limits_{(L)} P\,dx+Q\,dy+R\,dz \tag{12}$$

taken along the simple closed contour (L), *lying in* (T).

However, for it to be possible for us to use Stokes's formula, it is necessary in advance to impose on the three-dimensional domain (T), which we are considering, a natural restriction. In fact, we require that *for any simple piece-wise smooth closed contour* (L) *in the domain* (T) *we can "stretch" over this contour a piece-wise smooth (not self-intersecting) surface* (S), *which has* (L) *as its contour and is entirely contained in* (T). This property is analogous to the property

of being simply connected for a plane domain; when it holds, the space domain (T) is also called ("superficially"†) simply connected. We mention, by way of example, that the solid bounded by two concentric spherical surfaces will in this sense be simply connected, but a torus will not.

Let the domain (T) be (superficially) simply connected. After, so to speak, stretching the surface (S) over the contour (L), using Stokes's formula, we replace the curvilinear integral (12) by the surface integral

$$\iint_{(S)} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\,dx.$$

For this to vanish, clearly, the conditions

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \qquad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \tag{B}$$

are sufficient. At the same time it is easy to ascertain (as in Sec. 348) that these conditions are necessary, if we consider the plane figures (S), lying on the planes parallel to one or other of the coordinate planes in turn.

The reader can see that here we used Stokes's formula in exactly the same way as we used Green's formula in Sec. 348 for an analogous purpose.

It is easy to show that *the same conditions* (B) *will be necessary and sufficient for the integral*

$$\int_{(AB)} P\,dx + Q\,dy + R\,dz \tag{13}$$

not to depend on the form of the curve (AB), *joining any two points* A *and* B *of the domain* (T), with the hypothesis, of course, that this domain is (superficially) simply connected.

The necessity is established in the same way as in the plane case [Sec. 349]. As for the sufficiency, now, as then, it is easy to deal with the cases when the two curves (AIB) and $(AIIB)$, joining the points A and B, have no points in common except these. If, however,

† As distinct from another type of simple connectedness for a space domain, which we will discuss below [Sec. 381].

this is not so, and the curves intersect, then here the problem is more simple than in the plane case: in a connected space domain (T) we can always take a third such curve $(AIIIB)$, which does not intersect either of the previous ones†. Then

$$\int_{(AIB)} = \int_{(AIIIB)}, \qquad \int_{(AIIB)} = \int_{(AIIIB)},$$

hence it follows that always

$$\int_{(AIB)} = \int_{(AIIB)}.$$

With this investigation we can connect *the question of whether the differential expression*

$$P\,dx + Q\,dy + R\,dz \tag{14}$$

will be a (complete) differential of some function of the three variables. The necessity of the conditions (B) for this to be so can be verified

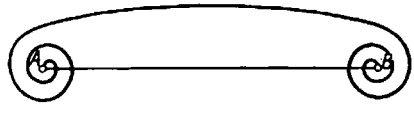

FIG. 57.

immediately, as in Sec. 350. The sufficiency is proved, also as in the plane case, by the immediate construction of the primitive—in the form of a curvilinear integral

$$F(x, y, z) = \int_{(x_0, y_0, z_0)}^{(x, y, z)} P\,dx + Q\,dy + R\,dz$$

[cf. (8) in Sec. 350], which, when conditions (B) are observed, does not depend on the path. Thus, for a domain (T) of the given type *the conditions* (B) *are necessary and sufficient for the expression* (14) *to be a complete differential.*

† As Fig. 57 shows, this cannot always be done in the plane case.

CHAPTER 23

TRIPLE INTEGRALS

§ 1. A triple integral and its evaluation

375. The problem of calculating the mass of a solid. Let there be given some solid (V), filled with matter, and at each of its points $M(x, y, z)$ let there be a known density

$$\varrho = \varrho(M) = \varrho(x, y, z)$$

of distribution of this matter. It is required to determine the whole mass m of the solid.

For the solution of this problem we divide up the solid (V) into a series of parts:

$$(V_1), (V_2), \ldots, (V_n)$$

and we choose within the limits of each of them a point

$$M_i(\xi_i, \eta_i, \zeta_i).$$

We assume, approximately, that within the limits of the part (V_i) the density is constant and is precisely equal to $\varrho(\xi_i, \eta_i, \zeta_i)$, the density at the chosen point. Then the mass m_i of this part can be expressed approximately thus:

$$m_i \doteq \varrho(\xi_i, \eta_i, \zeta_i) V_i,$$

and the mass of the whole solid will be

$$m \doteq \sum_{i=1}^{n} \varrho(\xi_i, \eta_i, \zeta_i) V_i.$$

If the diameters of all the parts tend to zero, then as limits are taken this approximate equation becomes exact, so that

$$m = \lim \sum_{i=1}^{n} \varrho(\xi_i, \eta_i, \zeta_i) V_i, \tag{1}$$

and the problem is solved.

We see that the solution of the problem reduces to the consideration of the limit of a particular sum—a type of one of the integral sums of various kinds with which we have dealt repeatedly throughout this whole course.

We often need to consider a similar kind of limit in mechanics and physics; such integrals have been given the name *triple integrals.* In the notation accepted for them the result obtained above can be written in the form:

$$m = \iiint\limits_{(V)} \varrho(x, y, z)\, dV. \tag{2}$$

The present chapter is devoted, basically, to the theory of triple integrals and their important applications. Because a whole series of propositions established for double integrals carries over, together with their proofs, to the case of triple integrals, we will usually consider it sufficient simply to formulate these propositions, leaving it to the reader to rewrite the previous proofs.

376. A triple integral and the conditions for its existence. In the construction of a general definition for the new integral form, the triple integral, a fundamental part is played by the concept of the volume of a solid, in the same way as the area of a plane figure forms the foundation for the definition of a double integral.

We are already familiar with the concept of a volume and we have often encountered it. The conditions for the existence of the volume for a given solid are contained in the fact that the surface bounding it must have volume equal to zero [Sec. 197]. We will only consider such surfaces as these, so that the existence of the volumes in all the cases we need is guaranteed. We note that, in particular, the smooth and piece-wise smooth surfaces belong to this class of surface.

Now in some space domain (V) let there be given a function $f(x, y, z)$. We divide up this domain by means of a net of surfaces into a finite number of parts $(V_1), (V_2), \ldots, (V_n)$, which have the volumes $V_1, V_2, \ldots, V_n$, respectively. Within the limits of the ith element (V_i) we choose an arbitrary point (ξ_i, η_i, ζ_i), multiply the value of the function at this point $f(\xi_i, \eta_i, \zeta_i)$ by the volume V_i and form the integral sum

$$\sigma = \sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i) V_i.$$

The limit I of this sum, as the greatest of the diameters of all the domains (V_i) tend to zero, is called the triple integral of the function $f(x, y, z)$ in the domain (V). It is denoted by the symbol

$$I = \iiint_{(V)} f(x, y, z)\, dV = \iiint_{(V)} f(x, y, z)\, dx\, dy\, dz.$$

A finite limit for a form like this can only exist for a bounded function; for such a function we also introduce, besides the integral sum σ, the Darboux sums:

$$s = \sum_{i=1}^{n} m_i V_i, \qquad S = \sum_{i=1}^{n} M_i V_i,$$

where

$$m_i = \inf_{(V_i)} \{f\}, \qquad M_i = \sup_{(V_i)} \{f\}.$$

In the usual way we establish that for the integral to exist it is necessary and sufficient to have the condition

$$\lim (S - s) = 0$$

or

$$\lim \sum_{i=1}^{n} \omega_i V_i = 0,$$

where $\omega_i = M_i - m_i$ is the oscillation of the function f in the domain (V_i). [We note that when the integral exists both the sums s and S have it as their limits.]

From this it follows immediately that *every continuous function f is integrable.*

We can extend these conditions somewhat, and in fact: *every bounded function, all of whose discontinuities lie on a finite number of surfaces with zero volume, is integrable.* [Cf. Secs. 339 and 340.]

377. The properties of integrable functions and triple integrals. It is sufficient to enumerate these properties [they are proved analogously to those given in Sec. 341].

(1) The existence and magnitude of a triple integral does not depend on the values assumed by the function along a finite number of surfaces with zero volume.

(2) If $(V) = (V') + (V'')$, *then*

$$\iiint_{(V)} f\,dV = \iiint_{(V')} f\,dV + \iiint_{(V'')} f\,dV,$$

where from the existence of the integral on the left follows the existence of the integrals on the right, and conversely.

(3) If $k = const$, *then*

$$\iiint_{(V)} kf\,dV = k\iiint_{(V)} f\,dV,$$

where from the existence of the integral on the right follows the existence of the integral on the left.

(4) If two functions f and g are integrable in the domain (V), *then the functions* $f \pm g$ *are also integrable, where*

$$\iiint_{(V)} (f \pm g)\,dV = \iiint_{(V)} f\,dV \pm \iiint_{(V)} g\,dV.$$

(5) If for the functions f and g, integrable in the domain (V), *the inequality* $f \leqslant g$ *is satisfied, then*

$$\iiint_{(V)} f\,dV \leqslant \iiint_{(V)} g\,dV.$$

(6) When the function f is integrable, then the function $|f|$ *is also integrable, and the inequality*

$$\left|\iiint_{(V)} f\,dV\right| \leqslant \iiint_{(V)} |f|\,dV.$$

holds.

(7) If the function f integrable in (V) *satisfies the inequality*

$$m \leqslant f \leqslant M,$$

then

$$mV \leqslant \iiint_{(V)} f\,dV \leqslant MV.$$

In other words, the mean value theorem

$$\iiint_{(V)} f\,dV = \mu V \qquad (m \leqslant \mu \leqslant M)$$

holds.

When the function f is continuous this formula can be rewritten in the form

$$\iiint_{(V)} f\,dV = f(\bar{x}, \bar{y}, \bar{z})V, \tag{3}$$

where $(\bar{x}, \bar{y}, \bar{z})$ is some point of the domain (V).

Further, it is easy to extend to the three-dimensional case the contents of Sec. 342: in the same way as there, we establish the concept of a function of the (three-dimensional) domain, in particular, an additive function. An important example of such a function (see *(2)*) is the integral with respect to a variable domain (v):

$$\Phi((v)) = \iiint_{(v)} f\,dv. \tag{4}$$

We introduce, by analogy with our previous reasoning, the concept of an arbitrary function $\Phi((v))$ over the domain at a given point M; this is what we call the limit

$$\lim_{(v)\to M} \frac{\Phi((v))}{v}$$

as the domain (v) containing it shrinks to the point M.

(8) If the function under the integral is continuous, then the derivat ve of the integral (4) *over the domain at the point* $M(x, y, z)$ *will have precisely the same value as the function under the integral at this point, i.e.* $f(M) = f(x, y, z)$.

Thus, with the given hypothesis the integral (4) is in a certain sense a "primitive" for the function f.

378. The evaluation of a triple integral. Here again the matter reduces to the evaluation of repeated integrals, composed of integrals of lower multiplicity. We presuppose that the function $f(x, y, z)$ is continuous in the domain under consideration, which guarantees the existence of all integrals which occur below. We consider first the case when the solid, over which the function $f(x, y, z)$ is integrated, is a rectangular parallelepiped

$$(T) = [a, b;\ c, d;\ e, f],$$

which projects onto the rectangle

$$(R) = [c, d;\ e, f]$$

on the yz plane.

Then we have first of all

$$\iiint_{(T)} f(x, y, z)\,dT = \int_a^b dx \int_{(R)} f(x, y, z)\,dR. \tag{5}$$

By replacing the double integral by a repeated one, we finally reduce the evaluation of the triple integral to the successive evaluation of three simple integrals:

$$\iiint_{(T)} f(x, y, z)\,dT = \int_a^b dx \int_c^d dy \int_e^f f(x, y, z)\,dz. \tag{6}$$

However, if the first two integrals are joined together in a double integral, then we obtain

$$\iiint_{(T)} f(x, y, z)\,dT = \iint_{(Q)} dx\,dy \int_e^f f(x, y, z)\,dz, \tag{7}$$

where $(Q) = [a, b; c, d]$. In all these cases, as is self-explanatory, the parts played by the variables x, y, z can be permuted.

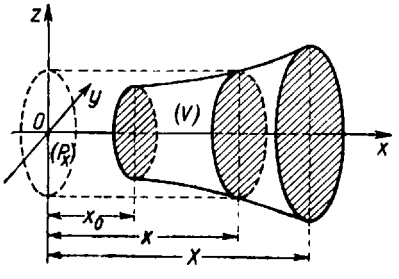

FIG. 58.

Now we presuppose that the triple integral is taken over a solid (V), which is not a rectangular parallelepiped. Let this solid be contained between the planes $x = x_0$ and $x = X$, and let every plane parallel to them, corresponding to a fixed value of $x (x_0 \leqslant x \leqslant X)$, intersect it in some figure which has an area; we denote its projection on the yz plane by (P_x) (Fig. 58). Then

$$\iiint_{(V)} f(x, y, z)\,dV = \int_{x_0}^X dx \iint_{(P_x)} f(x, y, z)\,dy\,dz. \tag{5a}$$

This is analogous to formula (5).

Let, further, the solid (V) be a "cylindrical block", bounded below and above, respectively, by the surfaces

$$z = z_0(x, y) \quad \text{and} \quad z = Z(x, y),$$

which are projected on the xy plane into the figure (D), bounded by the curve (K) with area equal to zero; the solid (V) is bounded laterally by a cylindrical surface with generators parallel to the z-axis, and with the curve (K) as a directrix (Fig. 59). Then, by analogy with formula (7), we have

$$\iiint_{(V)} f(x, y, z)\,dV = \iint_{(D)} dx\,dy \int_{z_0(x,y)}^{Z(x,y)} f(x, y, z)\,dz. \tag{7a}$$

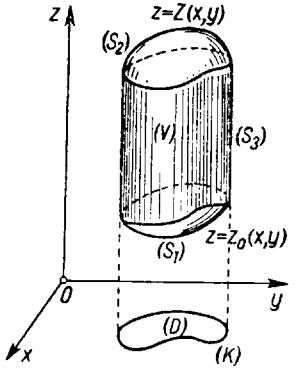

FIG. 59.

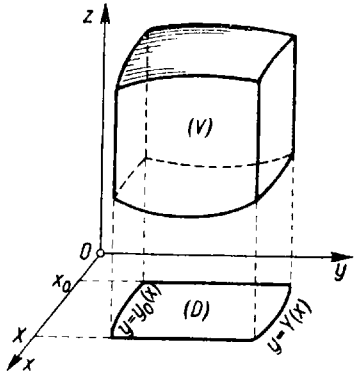

FIG. 60.

If the domain (D) is a curvilinear trapezoid, bounded by the two curves (Fig. 60)

$$y = y_0(x) \quad \text{and} \quad y = Y(x) \quad (x_0 \leqslant x \leqslant X)$$

and the straight lines $x = x_0$, $x = X$, then the solid (V) approaches both the types considered above. Replacing the double integral—either in the formula (5a) or in the formula (7a)—by a repeated integral, we obtain

$$\iiint_{(V)} f(x, y, z)\,dV = \int_{x_0}^{X} dx \int_{y_0(x)}^{Y(x)} dy \int_{z_0(x,y)}^{Z(x,y)} f(x, y, z)\,dz. \tag{6a}$$

This formula generalizes formula (6).

As in the simplest case, which was considered above, so here also we can permute the variables x, y, z.

Examples. (1) To evaluate the integral

$$I = \iiint \frac{dx\,dy\,dz}{(1+x+y+z)^3},$$

extended over the tetrahedron (V), bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x+y+z = 1$ (Fig. 61).

Solution. The projection of the solid on the xy plane is the triangle formed by the straight lines $x = 0$, $y = 0$ and $x+y = 1$. It is clear that the limits of variation of x are the numbers 0 and 1, and for a constant x within these limits the variable y ranges from 0 to $1-x$. If we fix both x and y, then the point can be moved along a vertical to the plane $z = 0$ as far as the plane $x+y+z = 1$; thus, the limits of variation of z will be 0 and $1-x-y$.

By formula (6a) we have

$$I = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} \frac{dz}{(1+x+y+z)^3}.$$

We evaluate the integrals successively, starting with the inner one:

$$\int_0^{1-x-y} \frac{dz}{(1+x+y+z)^3} = \frac{1}{2}\left[\frac{1}{(1+x+y)^2} - \frac{1}{4}\right];$$

$$\frac{1}{2}\int_0^{1-x}\left[\frac{1}{(1+x+y)^2} - \frac{1}{4}\right]dy = \frac{1}{2}\left(\frac{1}{x+1} - \frac{3-x}{4}\right),$$

finally

$$I = \frac{1}{2}\int_0^1 \left(\frac{1}{x+1} - \frac{3-x}{4}\right)dx = \frac{1}{2}\left(\log 2 - \frac{5}{8}\right).$$

(2) To evaluate the Dirichlet integral

$$D = \iiint\limits_{\substack{x,y,z \geqslant 0 \\ x+y+z \leqslant 1}} x^{p-1}y^{q-1}z^{r-1}\,dx\,dy\,dz,$$

presupposing that $p, q, r \geqslant 1$ (Fig. 61).

If we use a formula of the form (5a), putting the integration with respect to z in place of the integration with respect to x, then we obtain:

$$D = \int_0^1 z^{r-1}\,dz \iint\limits_{\substack{x,y \geqslant 0 \\ x+y \leqslant 1-z}} x^{p-1}y^{q-1}\,dx\,dy.$$

By the change of variables $x = (1-z)\xi$, $y = (1-z)\eta$ we reduce the double integral to one we already know [Sec. 344, *(3)*]:

$$(1-z)^{p+q} \iint\limits_{\substack{\xi,\eta \geqslant 0 \\ \xi+\eta \leqslant 1}} \xi^{p-1}\eta^{q-1}\,d\xi\,d\eta = (1-z)^{p+q}\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q+1)},$$

so that

$$D = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q+1)} \int_0^1 z^{r-1}(1-z)^{p+q}\,dz.$$

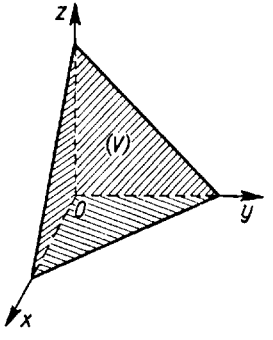

FIG. 61.

Finally, by using the well-known expression for the function B in terms of Γ [Sec. 311, (12)], we finally obtain:

$$D = \frac{\Gamma(p)\Gamma(q)\Gamma(r)}{\Gamma(p+q+r+1)}$$

—a result exactly analogous to that obtained in Sec. 344, *(3)*. In fact, it also holds under a more general hypothesis, namely: $p > 0$, $q > 0$, $r > 0$. But when the function under the integral becomes infinite (for instance, when $0 < p < 1$ this happens on the plane $x = 0$) the integral is "improper" and requires an additional passage to a limit. The definition of such an integral is constructed in exactly the same way as in the case of a double integral [see Sec. 344, *(4)*, Remark].

379. Mechanical applications. It is natural that all the geometrical and mechanical quantities connected with a distribution of matter within the limits of some solid (V) in space are expressed in principle this time by triple integrals, extended over the solid (V). Here it is simplest of all to use the principle of summation of infinitely small elements [cf. Secs. 204–208 and 345].

We denote by ϱ the density of the distribution of the matter at an arbitrary point of the solid (V); it is a function of the coordinates of the point; we will

always presuppose this function continuous. By summing the elements of mass $dm = \varrho\, dV = \varrho\, dx\, dy\, dz$, we will have for the value of the whole mass

$$m = \iiint\limits_{(V)} \varrho\, dV = \iiint\limits_{(V)} \varrho\, dx\, dy\, dz \tag{8}$$

[cf. Sec. 375].

Starting from the elementary moments

$$dK_{yz} = x\, dm = x\varrho\, dV, \quad dK_{zx} = y\, dm = y\varrho\, dV,$$

$$dK_{xy} = z\, dm = z\varrho\, dV,$$

we find the actual moments:

$$K_{yz} = \iiint\limits_{(V)} x\varrho\, dV, \quad K_{zx} = \iiint\limits_{(V)} y\varrho\, dV, \quad K_{xy} = \iiint\limits_{(V)} z\varrho\, dV, \tag{9}$$

and from them we find the coordinates of the centre of gravity:

$$\xi = \frac{\iiint\limits_{(V)} x\varrho\, dV}{m}, \quad \eta = \frac{\iiint\limits_{(V)} y\varrho\, dV}{m}, \quad \zeta = \frac{\iiint\limits_{(V)} z\varrho\, dV}{m}. \tag{10}$$

In the case of a homogeneous solid, $\varrho = \text{const}$, we obtain simply:

$$\xi = \frac{\iiint\limits_{(V)} x\, dV}{V}, \quad \eta = \frac{\iiint\limits_{(V)} y\, dV}{V}, \quad \zeta = \frac{\iiint\limits_{(V)} z\, dV}{V}.$$

The formulae for the moments of inertia with respect to the coordinate axes:

$$I_x = \iiint\limits_{(V)} (y^2 + z^2)\varrho\, dV, \quad I_y = \iiint\limits_{(V)} (z^2 + x^2)\varrho\, dV,$$

$$I_z = \iiint\limits_{(V)} (x^2 + y^2)\varrho\, dV \tag{11}$$

or with respect to the coordinate planes:

$$I_{yz} = \iiint\limits_{(V)} x^2\varrho\, dV, \quad I_{zx} = \iiint\limits_{(V)} y^2\varrho\, dV, \quad I_{xy} = \iiint\limits_{(V)} z^2\varrho\, dV \tag{12}$$

are also self-evident.

Finally, let the matter filling the solid (V) attract a point $A(\xi, \eta, \zeta)$ (of mass 1) by Newton's law (Fig. 62). The force of attraction due to the element of mass $dm = \varrho\, dV$ has the projections on the coordinate axes†

$$dF_x = \frac{x - \xi}{r^3}\varrho\, dV, \quad dF_y = \frac{y - \eta}{r^3}\varrho\, dV,$$

$$dF_z = \frac{z - \zeta}{r^3}\varrho\, dV,$$

† See the second footnote on p. 341.

where

$$r = \sqrt{[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]}$$

is the distance of the element (or the point at which we assume its mass to be concentrated) from the point A. Summing, we obtain for the projections of the complete force $\vec{F}$ of attraction on the coordinate axes

$$F_x = \iiint\limits_{(V)} \frac{x-\xi}{r^3}\varrho\, dV, \quad F_y = \iiint\limits_{(V)} \frac{y-\eta}{r^3}\varrho\, dV, \quad F_z = \iiint\limits_{(V)} \frac{z-\zeta}{r^3}\varrho\, dV. \tag{13}$$

Analogously we define the potential of our solid at a point:

$$W = \iiint \frac{\varrho\, dV}{r}. \tag{14}$$

If the point A lies outside the solid, then all these integrals are proper. In this case we can differentiate the integral W with respect to any of the variables

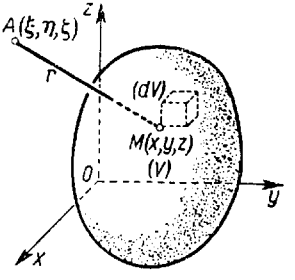

FIG. 62.

ξ, η, ζ under the sign of the integral on the basis of considerations similar to those which we used with respect to simple integrals [Sec. 297]. As a result we obtain that

$$\frac{\partial W}{\partial \xi} = F_x, \quad \frac{\partial W}{\partial \eta} = F_y, \quad \frac{\partial W}{\partial \zeta} = F_z. \tag{15}$$

In the case when the point A itself lies on the solid (V), $r = 0$ at this point and the functions under the integrals in (13) and (14) cease to be bounded near it. However, these integrals all exist, as "improper" integrals, and the fundamental relations (15) are satisfied for them.

Remark. In Sec. 345 for the statical moments of a homogeneous cylindrical block (when $\varrho = 1$) we had the formulae:

$$K_{yz} = \iint\limits_{(P)} zx\, dx\, dy, \quad K_{zx} = \iint\limits_{(P)} zy\, dx\, dy, \quad K_{xy} = \frac{1}{2}\iint\limits_{(P)} z^2\, dx\, dy.$$

These, of course, follow from the general formulae (9).

We have, for instance,

$$K_{xy} = \iiint\limits_{(V)} z\,dV = \iint\limits_{(P)} dx\,dy \int\limits_{0}^{z(x,y)} z\,dz;$$

but

$$\int\limits_{0}^{z(x,y)} z\,dz = \frac{1}{2} z^2 \Big|_{z=0}^{z=z(x,y)},$$

which reduces to the required result.

In Sec. 345, instead of the evaluation of the last integral, facts from the field of mechanics were taken into consideration (with respect to the statical moment of an elementary column).

§ 2. Ostrogradski's formula

380. Ostrogradski's formula. In the theory of double integrals we encountered Green's formula, which connects a double integral over a plane domain with a curvilinear integral along the contour of the domain. Its analogue in the theory of triple integrals is Ostrogradski's formula, which connects a triple integral over a space domain with a surface integral over the boundary of the domain.

We will consider the solid (V) (Fig. 59), bounded by the smooth surfaces

$$\left.\begin{array}{ll} (S_1) & z = z_0(x, y) \\ (S_2) & z = Z(x, y) \end{array}\right\} (z_0 \leqslant Z)$$

and the cylindrical surface (S_3), whose generators are parallel to the z-axis. The directrix here will be the piece-wise smooth closed curve (K) (with zero area) on the xy plane, bounding the domain (D), which is the projection of the solid (V) on this plane.

Let us suppose that in the domain (V) there is defined some function $R(x, y, z)$, continuous together with its derivative $\partial R/\partial z$ in the whole domain (V), excluding its boundary. Then the formula

$$\iiint\limits_{(V)} \frac{\partial R}{\partial z}\,dx\,dy\,dz = \iint\limits_{(S)} R\,dx\,dy \tag{1}$$

holds, where (S) is the surface bounding the solid and the integral on the right is extended over its outer side.

In fact, by formula (7a), Sec. 378

$$\iiint\limits_{(V)} \frac{\partial R}{\partial z}\,dx\,dy\,dz = \iint\limits_{(D)} dx\,dy \int\limits_{z_0(x,\,y)}^{Z(x,\,y)} \frac{\partial R}{\partial z}\,dz$$

$$= \iint\limits_{(D)} R(x, y, Z(x, y))\,dx\,dy - \iint\limits_{(D)} R(x, y, z_0(x, y))\,dx\,dy.$$

If we take into consideration the surface integrals, then, on the strength of formulae (3) and (3a), Sec. 372,

$$\iiint\limits_{(V)} \frac{\partial R}{\partial z}\,dx\,dy\,dz = \iint\limits_{(S_2)} R(x, y, z)\,dx\,dy + \iint\limits_{(S_1)} R(x, y, z)\,dx\,dy,$$

where the first of the integrals on the right is extended over the upper side of the surface (S_2), and the second is extended over the lower side (S_1). The equation is not violated if we add to its right side the integral

$$\iint\limits_{(S_3)} R(x, y, z)\,dx\,dy,$$

extended over the outer side of the surface (S_3), because this integral is equal to zero [Sec. 372, (5)]. Combining all three surface integrals into one, we again arrive at formula (1), which is a special case of Ostrogradski's formula.

It is easy to understand that formula (1) is true for a broader class of solids, which can be divided up into parts which are of the type studied. We can also prove that formula (1) is valid in general for solids bounded by arbitrary piece-wise smooth surfaces.

Analogously to formula (1) the formulae:

$$\iiint\limits_{(V)} \frac{\partial P}{\partial x}\,dx\,dy\,dz = \iint\limits_{(S)} P\,dy\,dz, \tag{2}$$

$$\iiint\limits_{(V)} \frac{\partial Q}{\partial y}\,dx\,dy\,dz = \iint\limits_{(S)} Q\,dz\,dx, \tag{3}$$

also hold, if the functions P and Q are continuous in the domain (V) together with their derivatives $\partial P/\partial x$ and $\partial Q/\partial y$.

Adding together all three formulae (1), (2), (3), we arrive at the general form of *Ostrogradski's formula*:

$$\iiint_{(V)} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\, dy\, dz = \iint_{(S)} P\, dy\, dz + Q\, dz\, dx + R\, dx\, dy. \qquad (4)$$

It expresses the general form of a surface integral of the second type, extended over the outside of a closed surface in terms of a triple integral taken over the solid bounded by this surface.

If we take into consideration surface integrals of the first type, then we obtain another very useful and easily remembered form of Ostrogradski's formula:

$$\iiint_{(V)} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\, dy\, dz = \iint_{(S)} (P\cos\lambda + Q\cos\mu + R\cos\nu)\, dS, \qquad (5)$$

where λ, μ, ν are the angles made by the outer normal to the surface (S) with the coordinate axes.

Remark I. Sometimes Ostrogradski's formula is connected with Gauss's name. Gauss considered only very special cases of this formula, where every time its derivative has to be given again. In the general form (4) this formula was first given in 1828 by Ostrogradski, who applied it to the problem of the propagation of heat in a rigid body.

Remark II. The formulae of Green, Stokes and Ostrogradski are connected by one idea: they express an integral extended over some geometrical form in terms of an integral taken over the boundary of this form. Here Green's formula refers to the case of two-dimensional space; Stokes's formula also refers to the case of two-dimensional, but "curved", space, and Ostrogradski's formula refers to the case of three-dimensional space.

In the fundamental formula of integral calculus

$$\int_a^b f'(x)\, dx = f(b) - f(a)$$

we can also see a sort of analogy with these formulae—for one-dimensional space.

381. Some examples of applications of Ostrogradski's formula. *(1) The representation of the volume of a solid in terms of surface integrals.* Analogously with Sec. 347, we can choose functions P, Q, R in different ways in formula (4) so that the expression under the integral sign in the triple integral is equal to 1, so that this integral reduces to the volume V of the solid (V). Thus, the volume V is represented in the form of a surface integral, extended over the surface (S) bounding the solid (V). Thus, in (4) letting in turn

$$P = x,\ Q = 0,\ R = 0; \quad P = 0,\ Q = y,\ R = 0; \quad P = 0,\ Q = 0,\ R = z,$$

we arrive at the formulae:

$$V = \iint\limits_{(S)} x\,dy\,dz = \iint\limits_{(S)} y\,dz\,dx = \iint\limits_{(S)} z\,dx\,dy, \tag{6}$$

where all the integrals are taken over the outer side of the surface (S). The more symmetric formula, corresponding to

$$P = \tfrac{1}{3}x, \quad Q = \tfrac{1}{3}y, \quad R = \tfrac{1}{3}z$$

is convenient; it has the form

$$V = \tfrac{1}{3}\iint\limits_{(S)} x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$$

or, if we go over to an integral of the first type,

$$V = \tfrac{1}{3}\iint\limits_{(S)} (x\cos\lambda + y\cos\mu + z\cos\nu)\,dS$$

(here $\cos\lambda$, $\cos\mu$, $\cos\nu$ denote the direction cosines of the outer normal $\vec{n}$ to surface).

We can represent this formula in yet another way: if we consider the vector $\vec{r}$, joining the origin to a variable point on the surface, with projections x, y, z on the coordinate axes, then the expression in the brackets can be rewritten in the form

$$r\cos(\vec{r},\ \vec{n})$$

and finally

$$V = \tfrac{1}{3}\iint\limits_{(S)} r\cos(\vec{r},\ \vec{n})\,dS.$$

It was in this form that Gauss discovered this result in 1813.

(2) Equilibrium of a rigid closed surface. We will prove that a rigid closed surface subject to a uniform pressure remains in equilibrium.

For this purpose we will establish that *the principal vector* and *the principal moment* (with respect to any point) of the whole system of forces acting on the surface are equal to zero.

We single out an element (dS) of the surface. If we denote by $p = \text{const}$ the pressure, i.e. the force, acting on unit area, then the elementary force acting on (dS) along the normal to this element will have the projections on the axes

$$-p\cos\lambda\, dS, \quad -p\cos\mu\, dS, \quad -p\cos\nu\, dS \tag{7}$$

(the minus sign occurs because the pressure is directed towards the inside of the surface, and λ, μ, ν are the angles made by the outer normal with the coordinate axes).

The projections R_x, R_y, R_z of the principal vector are obtained from the projections (7) of the elementary forces by summation:

$$R_x = -p\iint\limits_{(S)}\cos\lambda\, dS, \quad R_y = -p\iint\limits_{(S)}\cos\mu\, dS, \quad R_z = -p\iint\limits_{(S)}\cos\nu\, dS.$$

But all these integrals are equal to zero, which is evident from Ostrogradski's formula, if we set in it

$$P = 1,\ Q = R = 0; \quad Q = 1,\ P = R = 0; \quad R = 1,\ P = Q = 0.$$

Thus, *the principal vector of the pressure is equal to zero.*

To define the principal moment of a system of elementary forces, say, with respect to the origin of coordinates, we sum the projections on the axes of the moments of these elementary forces:

$$p(z\cos\mu - y\cos\nu)\, dS,\ p(x\cos\nu - z\cos\lambda)\, dS,\ p(y\cos\lambda - x\cos\mu)\, dS\dagger.$$

Thus, the projections of the principal moment of the pressure with respect to the origin will be:

$$L_x = p\iint\limits_{(S)}(z\cos\mu - y\cos\nu)\, dS, \quad L_y = p\iint\limits_{(S)}(x\cos\nu - z\cos\lambda)\, dS,$$

$$L_z = p\iint\limits_{(S)}(y\cos\lambda - x\cos\mu)\, dS.$$

if in Ostrogradski's formula we take $P = 0$, $Q = pz$, $R = -py$, then we have that $L_x = 0$. In the same way it is easy to establish that $L_y = L_z = 0$. *The principal moment of the pressure* (with respect to the origin) *is equal to zero.* This completes the proof.

(3) Archimedes' law. It is known that the pressure of a fluid on an area immersed in it is directed along the normal to the area and is equal to the weight of the column of fluid, which has this area as its base, and whose height is the depth of immersion of the area. Now let us suppose that a rigid body (V) is immersed in the fluid; on each element (dS) of its surface (S) the fluid exerts a pressure according to the law quoted. It is required to find the resultant of the elementary pressures and its point of application.

---

† We recall that if summands of the force along the axes are X, Y, Z and if the resultant acts at the point (x, y, z), then the moment of the force with respect to the point (ξ, η, ζ) has the following projections on the axes:

$$L_x = (y-\eta)Z - (z-\zeta)Y, \quad L_y = (z-\zeta)X - (x-\xi)Z,$$

$$L_z = (x-\xi)Y - (y-\eta)X.$$

To solve this problem we choose a coordinate system consisting of the xy plane on the free surface of the liquid, and the z-axis directed vertically downwards.

Let the specific weight of the fluid be equal to ϱ, and let the depth of immersion of the element (dS) be z; then the pressure experienced by this element will be

$$\varrho z\, dS,$$

and its projections on the axes will be:

$$-\varrho z \cos\lambda\, dS, \quad -\varrho z \cos\mu\, dS, \quad -\varrho z \cos\nu\, dS.$$

In this case for the projections of the principal vector on the axes we have:

$$R_x = -\varrho \iint\limits_{(S)} z \cos\lambda\, dS, \quad R_y = -\varrho \iint\limits_{(S)} z \cos\mu\, dS,$$

$$R_z = -\varrho \iint\limits_{(S)} z \cos\nu\, dS.$$

Using Ostrogradski's formula, as in the preceding problem, it is easy to obtain

$$R_x = R_y = 0, \quad R_z = -\varrho \iiint\limits_{(V)} dV = -\varrho V.$$

Thus, *the principal vector of the pressure is directed vertically upwards and is equal to the weight of the amount of liquid displaced.*

Now we consider the moments of the elementary forces with respect to the centre of gravity $C(\xi, \eta, \zeta)$ of the solid (here and in what follows we mean the centre of gravity of a *geometrical* body with uniform distribution of mass; it may not coincide with the centre of gravity of the physical body). The components of the elementary moments along the axes will be

$$\varrho z[(z-\zeta)\cos\mu - (y-\eta)\cos\nu]\, dS, \quad \varrho z[(x-\xi)\cos\nu - (z-\zeta)\cos\lambda]\, dS,$$
$$\varrho z[(y-\eta)\cos\lambda - (x-\xi)\cos\mu]\, dS,$$

and for the components of the principal moment (with respect to the point C) we obtain:

$$L_x = \varrho \iint\limits_{(S)} z[(z-\zeta)\cos\mu - (y-\eta)\cos\nu]\, dS,$$

$$L_y = \varrho \iint\limits_{(S)} z[(x-\xi)\cos\nu - (z-\zeta)\cos\lambda)]\, dS,$$

$$L_z = \varrho \iint\limits_{(S)} z[(y-\eta)\cos\lambda - (x-\xi)\cos\mu]\, dS$$

Applying Ostrogradski's formula to the first integral, we find that

$$L_x = \varrho \iiint\limits_{(V)} \left[\frac{\partial z(z-\zeta)}{\partial y} - \frac{\partial z(y-\eta)}{\partial z}\right] dV$$

$$= \varrho \iiint\limits_{(V)} (\eta - y)\, dV = \varrho \left[\eta V - \iiint\limits_{(V)} y\, dV\right] = 0,$$

because the integral $\iiint\limits_{(V)} y\,dV$ is the statical moment of the body with respect to the xz plane and is equal to ηV. Analogously we establish that $L_y = 0$; from this we see that $L_z = 0$ also.

Thus, *the principal moment of the pressure with respect to the centre of gravity of the body is equal to zero.* Comparing this assertion with the proposition proved earlier about the principal vector, we arrive at the conclusion: *the force acting on a body immersed in a fluid, and due to the fluid, is equal to the weight of the fluid displaced by the body; this force acts at the centre of gravity of the (geometrical) body and is directed vertically upwards.*

(4) The investigation of surface integrals. Let there be given continuous functions P, Q, R in some open domain (T) of three-dimensional space. Taking any closed surface (S) lying in this domain and bounding some body, we consider the surface integral

$$\iint\limits_{(S)} P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy = \iint\limits_{(S)} (P\cos\lambda + Q\cos\mu + R\cos\nu)\,dS. \qquad (8)$$

What condition must the functions P, Q, R satisfy in order that the integral (8) *shall always be equal to zero?*

This problem is analogous to the problem of the vanishing of the curvilinear integral along a closed contour [Secs. 348 and 374], which is easily solved with the help of Green's and Stokes's formulae. Here we have recourse to Ostrogradski's formula, presupposing, of course, that the derivatives of the functions P, Q, R which appear in this formula exist and are continuous.

However, in order to be justified in transforming the integral (8) by Ostrogradski's formula, it is also necessary in the present case to impose some restrictions on the fundamental domain (T). In fact, we require that, *provided that the simple closed surface (S) bounding the body (V) belongs to the domain (T), then this body also is completely contained in the domain (T).* A domain satisfying these conditions is called ("spacially") simply connected (cf. Sec. 374). The essence of this type of single connectedness lies in the absence of "holes", except at points; with respect to a body which does not extend to infinity we simply require that its boundary consists of one unique closed surface [cf. Sec. 348]. Therefore, for instance, as distinct from what was said concerning "superficial" simple connectedness in Sec. 374, here a torus will be a simply connected body, but a hollow sphere will not.

Ostrogradski's formula reduces at once to the desired condition:

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0. \qquad (B)$$

The sufficiency of this condition is obvious, and the necessity is easily proved by means of differentiating a triple integral over the domain [Sec. 377, *(8)*].

Analogous with the case of curvilinear integrals, the question of the vanishing of the integral over the closed surface is equivalent to the question of whether the integral over a non-closed surface, "stretched" over the given contour, is independent of the form of the surface. We will not stop to prove this.

§ 3. Change of variables in triple integrals

382. The transformation of space domains. The ideas developed in Sec. 352 in connection with the transformation of plane domains carry over in a natural way to the case of space domains.

Suppose that we have a space referred to the system of rectangular coordinates xyz, and another space with a system of coordinates $\xi\eta\zeta$. We consider two closed domains (D) and (Δ) in these spaces, bounded respectively by the surfaces (S) and (Σ), which we will always presuppose piece-wise smooth. Let us suppose that these domains are related to each other by a one-one continuous correspondence, which is defined by the formulae:

$$\left.\begin{aligned} x &= x(\xi, \eta, \zeta), \\ y &= y(\xi, \eta, \zeta), \\ z &= z(\xi, \eta, \zeta). \end{aligned}\right\} \tag{1}$$

In this, necessarily, *the points of the surface* (S) *correspond precisely to the points of the surface* (Σ), and conversely.

Let the functions (1) have continuous partial derivatives in the domain (Δ); then the functional determinant

$$\frac{D(x, y, z)}{D(\xi, \eta, \zeta)}$$

is also a continuous function in (Δ). Here [cf. Sec. 352] we will assume that *this determinant is always distinct from zero, and has a constant sign.*

Then the formulae (1) transform a piece-wise smooth surface contained in the domain (Δ) into a piece-wise smooth surface lying in (D), and conversely.

The symbols ξ, η, ζ which uniquely characterize the position of a point in the xyz space are called the *curvilinear coordinates* of this point. The points of the xyz space for which one of these coordinates keeps a constant value form a *coordinate surface.* In all there will

exist three families of such coordinate surfaces; through every point of the domain (D) passes one surface of each of these families.

However, all this will only be so under the hypothesis of the strictly one–one nature of the correspondence between the domains (D) and (Δ). In practice this one–one nature is often violated.

Examples. (1) The cylindrical coordinates combine the polar coordinates in the xy plane with the ordinary Cartesian z coordinate (Fig. 63). The formulae connecting them with the Cartesian coordinates have the form

$$x = \varrho\cos\theta, \quad y = \varrho\sin\theta, \quad z = z. \tag{2}$$

These formulae map the domain

$$0 \leqslant \varrho < +\infty, \quad 0 \leqslant \theta < 2\pi, \quad -\infty < z < +\infty$$

on the whole xyz space. We note, however, that the straight lines $\varrho = 0$, $z = z$ map onto the one point $(0, 0, z)$; this violates the one-one nature of the correspondence.

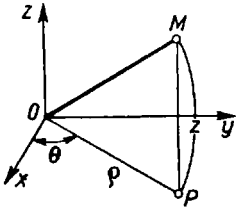

FIG. 63.

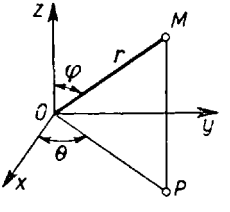

FIG. 64.

The coordinate surfaces in the case under consideration will be:

(a) $\varrho = \text{const}$—cylindrical surfaces with generators parallel to the z-axis; their directrices will be circles on the xy plane with centres at the origin;

(b) $\theta = \text{const}$—the semi-planes passing through the z-axis;

(c) $z = \text{const}$—planes parallel to the xy plane.

The functional determinant of the transformation is:

$$J = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -\varrho\sin\theta & \varrho\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \varrho \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \varrho.$$

Excluding the case $\varrho = 0$, it keeps the positive sign.

(2) The spherical coordinates, also called *polar coordinates in space*, are connected with the Cartesian coordinates by the formulae

$$x = r\sin\varphi\cos\theta, \quad y = r\sin\varphi\sin\theta, \quad z = r\cos\varphi,$$

where

$$0 \leqslant r < +\infty, \quad 0 \leqslant \varphi \leqslant \pi, \quad 0 \leqslant \theta < 2\pi.$$

The geometrical meaning of the quantities r, φ, θ is clear from Fig. 64: r is the radius vector OM, joining the origin (the pole) to the given point M, φ is the

angle made by this radius vector with the z-axis (the polar axis); θ is the angle made with the x-axis by the projection $OP = r\sin\varphi$ of the radius vector OM on the xy plane (perpendicular to the polar axis).

In this case we again encounter a violation of the one-one nature of the correspondence: the plane $r = 0$ in the space $r\varphi\theta$ maps into the origin of coordinates $x = y = z = 0$, the straight line $\varphi = 0(\pi)$, $r = r$ maps into the single point

$$x = y = 0, \quad z = r.$$

The coordinate surfaces form the three families:

(a) $r = \text{const}$—concentric spheres with centres at the origin of coordinates;

(b) $\varphi = \text{const}$—circular cones which have the z-axis as their axis;

(c) $\theta = \text{const}$—semi-planes passing through the z-axis.

The determinant of this transformation is:

$$J = \begin{vmatrix} \sin\varphi\cos\theta & \sin\varphi\sin\theta & \cos\varphi \\ r\cos\varphi\cos\theta & r\cos\varphi\sin\theta & -r\sin\varphi \\ -r\sin\varphi\sin\theta & r\sin\varphi\cos\theta & 0 \end{vmatrix} = r^2\sin\varphi.$$

It has the plus sign, after we exclude the case mentioned above, when $r = 0$, or $\varphi = 0(\pi)$ and the determinant vanishes.

383. An expression for volume in curvilinear coordinates. With the hypotheses and notation of Sec. 382 we consider the problem of expressing the volume of a (bounded) body (D) in the xyz space by means of a triple integral extended over the corresponding body (Δ) in the space $\xi\eta\zeta$†.

The desired volume is expressed first of all in terms of a surface integral of the second type [see Sec. 380, (6)]:

$$D = \iint\limits_{(S)} z\,dx\,dy,$$

extended over the outer side of the surface (S). We will try to transform this to an ordinary double integral.

We will start from the parametric equations of the surface (Σ):

$$\xi = \xi(u, v), \quad \eta = \eta(u, v), \quad \zeta = \zeta(u, v), \tag{3}$$

presupposing that the parameters vary in some domain (E) in the uv plane. In the transformation formulae (1) we replace ξ, η, ζ by

† As in Sec. 353, we presuppose here in addition the existence and continuity of the partial derivatives, say,

$$x''_{\xi\eta}, x''_{\eta\xi}, \ldots, y''_{\xi\eta}, y''_{\eta\xi}, \ldots;$$

this makes the proof easier, although it is not essential for the result itself.

the expressions in (3); then we obtain, clearly, parametric equations of the surface (S):

$$x = x\big(\xi(u,v),\ \eta(u,v),\ \zeta(u,v)\big) = x(u,v),$$
$$y = y(u,v), \quad z = z(u,v).$$

Letting

$$C = \frac{D(x,y)}{D(u,v)},$$

in formula (7), Sec. 372 [see the Remark] we have

$$D = \pm \iint_{(E)} zC\,du\,dv.$$

Because x, y depend on u, v by means of the variables ξ, η, ζ, then, by the well-known property of functional determinants [Sec. 326],

$$C = \frac{D(x,y)}{D(\xi,\eta)}\frac{D(\xi,\eta)}{D(u,v)} + \frac{D(x,y)}{D(\eta,\zeta)}\frac{D(\eta,\zeta)}{D(u,v)} + \frac{D(x,y)}{D(\zeta,\xi)}\frac{D(\zeta,\xi)}{D(u,v)}.$$

Substituting this expression for C in the integral obtained above, we find

$$D = \pm \iint_{(E)} z\left[\frac{D(x,y)}{D(\xi,\eta)}\frac{D(\xi,\eta)}{D(u,v)} + \frac{D(x,y)}{D(\eta,\zeta)}\frac{D(\eta,\zeta)}{D(u,v)} + \frac{D(x,y)}{D(\zeta,\xi)}\frac{D(\zeta,\xi)}{D(u,v)}\right] du\,dv. \quad (4)$$

Comparing this integral with the surface integral of the second type extended over the outer side of the surface (Σ):

$$\iint_{(\Sigma)} z\left[\frac{D(x,y)}{D(\xi,\eta)}\,d\xi\,d\eta + \frac{D(x,y)}{D(\eta,\zeta)}\,d\eta\,d\zeta + \frac{D(x,y)}{D(\zeta,\xi)}\,d\zeta\,d\xi\right]. \quad (5)$$

If we transform this, starting from the parametric equations (3), to an ordinary double integral, by means of a formula analogous to formula (7), Sec. 372, then we arrive at the integral (4). The only difference between these integrals lies in the sign.

Finally, from the integral (5) by means of Ostrogradski's formula we can go over to a triple integral over the domain (Δ):

$$D = \pm \iiint_{(\Delta)} \left\{\frac{\partial}{\partial\xi}\left[z\frac{D(x,y)}{D(\eta,\zeta)}\right] + \frac{\partial}{\partial\eta}\left[z\frac{D(x,y)}{D(\zeta,\xi)}\right] + \frac{\partial}{\partial\zeta}\left[z\frac{D(x,y)}{D(\xi,\eta)}\right]\right\} d\xi\,d\eta\,d\zeta.$$

The expression under the integral is equal to

$$\frac{\partial z}{\partial \xi}\frac{D(x,y)}{D(\eta,\zeta)}+\frac{\partial z}{\partial \eta}\frac{D(x,y)}{D(\zeta,\xi)}+\frac{\partial z}{\partial \zeta}\frac{D(x,y)}{D(\xi,\eta)}$$

$$+z\left[\frac{\partial}{\partial \xi}\frac{D(x,y)}{D(\eta,\zeta)}+\frac{\partial}{\partial \eta}\frac{D(x,y)}{D(\zeta,\xi)}+\frac{\partial}{\partial \zeta}\frac{D(x,y)}{D(\xi,\eta)}\right].$$

The sum which here lies in the first line is equal to the functional determinant

$$\frac{D(x,y,z)}{D(\xi,\eta,\zeta)}=\begin{vmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta}\\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta}\\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta}\end{vmatrix},$$

which it is easy to ascertain by expanding this determinant by the elements of its last row; the sum in the square brackets, as immediate calculation shows, is equal to zero†.

Thus, we arrive at the formula

$$D=\pm\iiint\limits_{(\Delta)}\frac{D(x,y,z)}{D(\xi,\eta,\zeta)}\,d\xi\,d\eta\,d\zeta.$$

If we recall that by hypothesis the functional determinant keeps the same sign, which it communicates to the integral, then it becomes clear (because we assume here $D>0$) that the sign before the integral

† Clearly,

$$\frac{\partial}{\partial \xi}\frac{D(x,y)}{D(\eta,\zeta)}=\frac{\partial^2 x}{\partial \eta\,\partial \xi}\frac{\partial y}{\partial \zeta}+\frac{\partial x}{\partial \eta}\frac{\partial^2 y}{\partial \zeta\,\partial \xi}-\frac{\partial^2 x}{\partial \zeta\,\partial \xi}\frac{\partial y}{\partial \eta}-\frac{\partial x}{\partial \zeta}\frac{\partial^2 y}{\partial \eta\,\partial \xi},$$

$$\frac{\partial}{\partial \eta}\frac{D(x,y)}{D(\zeta,\xi)}=\frac{\partial^2 x}{\partial \zeta\,\partial \eta}\frac{\partial y}{\partial \xi}+\frac{\partial x}{\partial \zeta}\frac{\partial^2 y}{\partial \xi\,\partial \eta}-\frac{\partial^2 x}{\partial \xi\,\partial \eta}\frac{\partial y}{\partial \zeta}-\frac{\partial x}{\partial \xi}\frac{\partial^2 y}{\partial \zeta\,\partial \eta},$$

$$\frac{\partial}{\partial \zeta}\frac{D(x,y)}{D(\xi,\eta)}=\frac{\partial^2 x}{\partial \xi\,\partial \zeta}\frac{\partial y}{\partial \eta}+\frac{\partial x}{\partial \xi}\frac{\partial^2 y}{\partial \eta\,\partial \zeta}-\frac{\partial^2 x}{\partial \eta\,\partial \zeta}\frac{\partial y}{\partial \xi}-\frac{\partial x}{\partial \eta}\frac{\partial^2 y}{\partial \xi\,\partial \zeta}.$$

Adding these equations term by term, we obtain that the right side is identically equal to zero.

coincides with the sign of the determinant. This gives us the right to rewrite the result obtained in the final form:

$$D = \iiint\limits_{(\Delta)} \left| \frac{D(x, y, z)}{D(\xi, \eta, \zeta)} \right| d\xi\, d\eta\, d\zeta \tag{6}$$

or, if, for simplicity, we denote the functional determinant by $J(\xi, \eta, \zeta)$:

$$D = \iiint\limits_{(\Delta)} |J(\xi, \eta, \zeta)|\, d\xi, d\eta, d\zeta. \tag{6*}$$

The expression under the integral

$$\left| \frac{D(x, y, z)}{D(\xi, \eta, \zeta)} \right| d\xi\, d\eta\, d\zeta = |J(\xi, \eta, \zeta)|\, d\xi\, d\eta\, d\zeta$$

is usually called the *element of volume in curvilinear coordinates.*

If we apply the mean value theorem to formula (6*), then we obtain the relation

$$D = |J(\bar{\xi}, \bar{\eta}, \bar{\zeta})|\, \Delta, \tag{7}$$

where $(\bar{\xi}, \bar{\eta}, \bar{\zeta})$ is some point in the domain (Δ), and Δ is the volume of this domain.

Hence it is easy to deduce that when the domain (Δ) is contracted to the point (ξ, η, ζ) we will have [cf. Sec. 377, *(8)*]:

$$|J(\xi, \eta, \zeta)| = \lim \frac{D}{\Delta},$$

so that the absolute value of the functional determinant is the coefficient of expansion of the $\xi\eta\zeta$ space (at a given point of it) in its transformation to the xyz space.

Remark. Formula (6) [(6*)] is derived under well-known hypotheses—a one-one and continuous correspondence between the domains (D) and (Δ) and so on. However, as in Sec. 354, *(4)*, we can show that the violation of these conditions at separate points or along separate lines and surfaces does not prevent the formula from being true, provided only that the functional determinant remains bounded.

384. A geometrical derivation. The derivation of formula (6) can also be carried out, following Ostrogradski, from purely geometrical considerations [cf. Sec. 355]. With the infinitely small rectangular parallelepiped in the $\xi\eta\zeta$ space with sides $d\xi, d\eta, d\zeta$ we compare

an elementary body in the xyz space between the coordinate surfaces "ξ" and "$\xi + d\xi$", "η" and "$\eta + d\eta$", "ζ" and "$\zeta + d\zeta$", which can be approximately regarded as a scalar parallelepiped. Its volume is equal to six times the volume of the tetrahedron with vertices at the points:

$$P_1(x, y, z), \quad P_2\left(x + \frac{\partial x}{\partial \xi} d\xi, \quad y + \frac{\partial y}{\partial \xi} d\xi, \quad z + \frac{\partial z}{\partial \xi} d\xi\right),$$

$$P_3\left(x + \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta, \quad y + \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta, \right.$$

$$\left. z + \frac{\partial z}{\partial \xi} d\xi + \frac{\partial z}{\partial \eta} d\eta\right),$$

$$P_4\left(x + \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta + \frac{\partial x}{\partial \zeta} d\zeta, \quad y + \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta + \frac{\partial y}{\partial \zeta} d\zeta, \right.$$

$$\left. z + \frac{\partial z}{\partial \xi} d\xi + \frac{\partial z}{\partial \eta} d\eta + \frac{\partial z}{\partial \zeta} d\zeta\right),$$

and by a well-known formula in analytical geometry is expressed (in absolute value) by the determinant

$$\begin{vmatrix} \frac{\partial x}{\partial \xi} d\xi & \frac{\partial x}{\partial \eta} d\eta & \frac{\partial x}{\partial \zeta} d\zeta \\ \frac{\partial y}{\partial \xi} d\xi & \frac{\partial y}{\partial \eta} d\eta & \frac{\partial y}{\partial \zeta} d\zeta \\ \frac{\partial z}{\partial \xi} d\xi & \frac{\partial z}{\partial \eta} d\eta & \frac{\partial z}{\partial \zeta} d\zeta \end{vmatrix} = \frac{D(x, y, z)}{D(\xi, \eta, \zeta)} d\xi\, d\eta\, d\zeta.$$

Summing these separate "elements of volume", we arrive at formula (6).

Thus, *the essence of the matter here is that to determine the volume of a body it is divided up into elements not by a system of mutually perpendicular planes, but by a net of coordinate surfaces.*

For an example in the case of cylindrical coordinates we consider an elementary domain (in the xyz space), bounded by two cylindrical surfaces of radii ϱ and $\varrho + d\varrho$, two horizontal planes lying at the heights z and $z + dz$, and two semiplanes passing through the z-axis and sloping at angles θ and $\theta + d\theta$ to the xz

plane (Fig. 65). By assuming that this domain is approximately a rectangular parallelepiped, we find without difficulty that its sides measure $d\varrho$, $\varrho d\theta$ and dz, so that the volume is equal to $\varrho d\varrho d\theta dz$, and the determinant representing the ratio of this volume to the volume $d\varrho d\theta dz$ of the elementary parallelepiped in the space $\varrho\theta z$ is equal to ϱ.

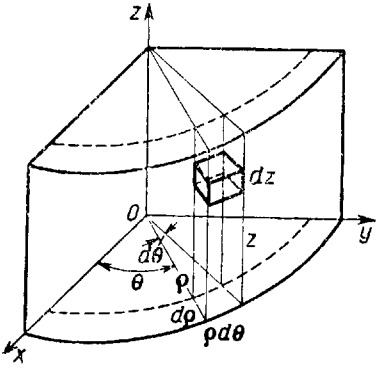

FIG. 65.

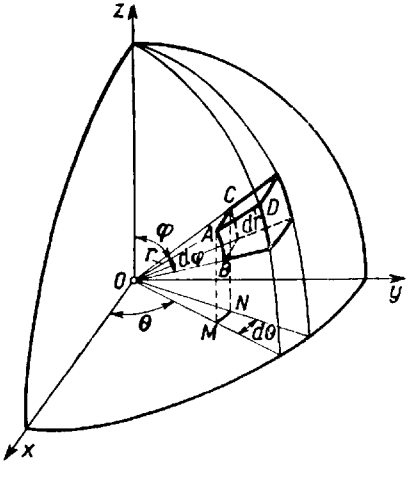

FIG. 66.

Analogously, in the case of spherical coordinates we consider the elementary domain (in the xyz space), bounded by the spheres of radii r and $r+dr$, the cones φ and $\varphi+d\varphi$ and the semi-planes θ and $\theta+d\theta$ (Fig. 66). This domain can also be taken as a rectangular parallelepiped with sides measuring $AD = dr$, $AB = r\,d\varphi$ and, finally, AC. Because the arc AC is equal to the projection MN,

and the latter is described by a radius $OM = r\sin\varphi$ and corresponds to a central angle $d\theta$, then $AC = r\sin\varphi\, d\theta$. By virtue of this the volume of the domain under consideration is equal to $r^2\sin\varphi\, dr\, d\varphi\, d\theta$, and the functional determinant is $r^2\sin\varphi$.

Both these results, found from elementary geometrical considerations, agree with what was said in Sec. 382, *(1)* and *(2)*.

385. Change of variables in triple integrals. By means of the expression for the volume in curvilinear coordinates it is not difficult to establish the general formula for the change of variables in triple integrals.

Between the domains (D) and (Δ) in the xyz and $\xi\eta\zeta$ spaces let there exist a correspondence, as characterized in Sec. 382. Assuming that all the conditions under which formula (6) was derived are satisfied, we now show that the following equation holds:

$$\iiint\limits_{(D)} f(x, y, z)\, dx\, dy\, dz$$

$$= \iiint\limits_{(\Delta)} f\big(x(\xi, \eta, \zeta),\ y(\xi, \eta, \zeta),\ z(\xi, \eta, \zeta)\big)\, |J(\xi, \eta, \zeta)|\, d\xi\, d\eta\, d\zeta, \quad (8)$$

$$\left(\text{where } J(\xi, \eta, \zeta) = \frac{D(x, y, z)}{D(\xi, \eta, \zeta)}\right),$$

which is completely analogous to the formula for the change of variables in double integrals. Here we presuppose that the function $f(x, y, z)$ is continuous. Thus both the integrals in equation (8) exist; we need only establish the equation itself.

For the proof we make the same suppositions as in Sec. 356. After dividing up the domains (D) and (Δ) by means of piecewise smooth surfaces into the elementary parts (D_i) and (Δ_i) $(i = 1, 2, \ldots, n)$ (which correspond to one another), we apply formula (7) to each pair of domains (D_i) and (Δ_i); we obtain

$$D_i = |J(\bar{\xi}_i, \bar{\eta}_i, \bar{\zeta}_i)|\, \Delta_i, \quad (9)$$

where $(\bar{\xi}_i, \bar{\eta}_i, \bar{\zeta}_i)$ is some point in the domain (Δ_i), which is independent of our choice. Let us take the corresponding point $(\bar{x}_i, \bar{y}_i, \bar{z}_i)$ in the domain (D_i), i.e. let us set

$$\bar{x}_i = x(\bar{\xi}_i, \bar{\eta}_i, \bar{\zeta}_i), \quad \bar{y}_i = y(\bar{\xi}_i, \bar{\eta}_i, \bar{\zeta}_i), \quad \bar{z}_i = z(\bar{\xi}_i, \bar{\eta}_i, \bar{\zeta}_i), \quad (10)$$

and we form the integral sum for the first of the integrals (8):

$$\sigma = \sum_i f(\bar{x}_i, \bar{y}_i, \bar{z}_i) D_i.$$

Substituting here the expressions (10) instead of $\bar{x}_i, \bar{y}_i, \bar{z}_i$, and the expression (9) instead of D_i, we arrive at the sum

$$\sigma = \sum_i f(x(\bar{\xi}_i, \bar{\eta}_i, \bar{\zeta}_i), \quad y(\bar{\xi}_i, \bar{\eta}_i, \bar{\zeta}_i), \quad z(\bar{\xi}_i, \bar{\eta}_i, \bar{\zeta}_i)) \, |J(\bar{\xi}_i, \bar{\eta}_i, \bar{\zeta}_i)| \Delta_i,$$

which, clearly, is the integral sum for the second of the integrals (8).

We make the diameters of the domains (Δ_i) tend to zero, as a result of which, by virtue of the continuity of the correspondence, the diameters of the domains (D_i) tend to zero. The sum σ must tend simultaneously to both the integrals, from which follows the required equation.

As in the case of double integrals, formula (8) holds in a series of cases when the hypotheses formulated above are violated at separate points or along a finite number of lines or surfaces.

386. Examples. *(1)* To calculate the volume of the body bounded by the surface

$$(x^2 + y^2 + z^2)^2 = a^3 z.$$

Solution. The body is situated symmetrically with respect to the yz and zx planes, because x and y only appear squared in the equation. Further, since the left side of the equation is always positive, it must necessarily be true that $z \geqslant 0$, i.e. the whole body lies above the xy plane. These remarks enable us to limit ourselves to the calculation of the volume of the quarter of the body lying in the first octant.

The presence of the expression $x^2 + y^2 + z^2$ in the equation suggests to us the transition to spherical coordinates. Substituting in the equation of the surface the expressions

$$x = r \sin\varphi \cos\theta, \quad y = r \sin\varphi \sin\theta, \quad z = r \cos\varphi,$$

we arrive at the equation of the surface in spherical coordinates:

$$r = a \sqrt[3]{\cos\varphi}.$$

Because the first octant is characterized by the inequalities

$$0 \leqslant \varphi \leqslant \frac{\pi}{2}, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2},$$

and taking into account the value of the functional determinant $J = r^2 \sin\varphi$ [Sec. 382, *(2)*], it follows that

$$V = 4\int_0^{\pi/2} d\theta \int_0^{\pi/2} d\varphi \int_0^{a\sqrt[3]{(\cos\varphi)}} r^2 \sin\varphi\, dr = \frac{2}{3}\pi a^3 \int_0^{\pi/2} \sin\varphi \cos\varphi\, d\varphi = \frac{1}{3}\pi a^3.$$

(2) The application of *cylindrical coordinates* to the evaluation of the volume of a body leads to an interesting formula.

We consider the body (V), bounded by a piece-wise smooth surface, and we presuppose that the semi-surface starting from the z-axis and corresponding to $\theta = \text{const}$ intersects the body in some plane figure (Q_θ) as θ varies from α to β (Fig. 67). Then

$$V = \iiint_{(V)} \varrho\, d\varrho\, d\theta\, dz = \int_\alpha^\beta d\theta \iint_{(Q_\theta)} \varrho\, d\varrho\, dz,$$

where the figure (Q_θ) is referred for convenience to the rectangular system of coordinates ϱz which rotate around the z-axis with the given semi-surface†.

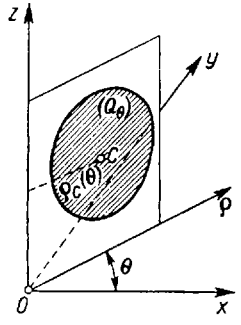

FIG. 67.

Now it is easy to see that the double integral $\iint_{(Q_\theta)} \varrho\, d\varrho\, dz$ represents the moment of the figure (Q_θ) with respect to the z-axis, which is equal to the product of the area $Q(\theta)$ of this figure by the distance $\varrho_C(\theta)$ of its centre of gravity C from the z-axis:

$$\iint_{(Q_\theta)} \varrho\, d\varrho\, dz = Q(\theta)\varrho_C(\theta).$$

Substituting this in the expression for the volume, we arrive at the final formula:

$$V = \int_\alpha^\beta Q(\theta)\varrho_C(\theta)\, d\theta.$$

† Instead of the figure with respect to the fixed plane ϱz in the space $\varrho\theta z$, which is identical with it.

This formula was indicated by P. P. Kuskov. It is particularly convenient for the definition of the volume of bodies obtained as a result of the helical motion of plane figures (constant or deforming), such as helical screws, springs, and so on.

If the body (V) is simply a solid of rotation of an invariable figure (Q) about the z-axis, which does not intersect it, then $Q = \text{const}$, $\varrho_C = \text{const}$, $\alpha = 0$, $\beta = 2\pi$, and the formula takes the form

$$V = Q \cdot 2\pi\varrho_C.$$

This expresses the well-known Guldini theorem, which states that *the volume of a solid of rotation of a plane figure around an axis which does not intersect it is equal to the product of the area of this figure by the length of the circumference described by the centre of gravity of the figure.* Thus, Kuskov's formula is a natural generalization of this classical theorem and, conversely, can easily be obtained from it.

(3) To find the attraction experienced by an arbitrary point A in space (of mass 1) which is due to a homogeneous sphere with density ϱ.

Let the radius of the sphere be equal to R, and the distance $OA = a$. We order the coordinate axes in such a way that the point A is located on the positive side of the z-axis. Then

$$F_z = \iiint\limits_{(V)} \frac{\varrho(z-a)}{[x^2+y^2+(z-a)^2]^{3/2}}\,dx\,dy\,dz.$$

Going over to spherical coordinates, we find

$$F_z = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \frac{\varrho r^2(r\cos\varphi - a)\sin\varphi\,dr\,d\varphi\,d\theta}{(r^2+a^2-2ar\cos\varphi)^{3/2}}.$$

But, in determining the attraction of a spherical layer [Sec. 370, *(1)*], we have already found the value of the double integral

$$\int_0^{2\pi}\int_0^{\pi} \frac{(r\cos\varphi - a)\sin\varphi\,d\varphi\,d\theta}{(r^2+a^2-2ar\cos\varphi)^{3/2}} = \begin{cases} 0 & \text{when } a < r \\ -4\pi/a^2 & \text{when } a > r. \end{cases}$$

In this case when $a > R$

$$F_z = -\frac{4\pi\varrho}{a^2}\int_0^R r^2\,dr = -\frac{4}{3}\pi R^3\varrho\frac{1}{a^2},$$

and when $a < R$

$$F_z = -\frac{4\pi\varrho}{a^2}\int_0^a r^2\,dr = -\frac{4}{3}\pi\varrho a.$$

At the same time, clearly, $F_x = F_y = 0$. Thus, in all cases the attraction is directed towards the centre of the sphere.

Here a point lying outside the sphere ($a > R$) *experiences the same attraction from the latter as it would experience if all the mass* $m = (4/3)\pi R^3\varrho$ *of the sphere were concentrated at the centre.* On the other hand, because, with respect to a point lying inside the sphere ($a < R$), the attraction does not depend on R and has the same value as in the case $R = a$, it is clear that *an external spherical layer has no effect on a point lying inside it.*

It is necessary to note that when $a > R$ (when the point A being attracted lies outside the sphere) the function under the integral is continuous, and the calculations do not need any explanation. The situation is different when $a \leqslant R$ (when the point A lies inside the sphere or on its surface). In this case, in the neighbourhood of the point A the function under the integral ceases to be bounded, and the integral must be regarded as improper. After a change of variables the singularity disappears; this fact enables us to establish the existence of the integral, and justifies the whole calculation.

387. Historical note. Triple integrals first appeared in a work of Lagrange in 1773 (published two years later), which was fundamentally devoted to the theory of attraction. Having defined for the force of attraction at a point the "elementary parallelepiped" $dxdydz$ of its projections on the three coordinate axes, Lagrange investigated their integral, which he presupposed "extended over all the points of the body". Here Lagrange gave a definition of a triple integral, which [as with Euler's double integral, see Sec. 359] was obtained by repeated integration; here it is clearly indicated between what limits each of the successive three simple integrals must be taken. At the same time, Lagrange considered the triple integral as a sum and even denoted it on other occasions by $\sum$.

Having ascertained that (even in the case of a sphere) the integration is difficult, Lagrange indicated that, to simplify it, he would thereafter use other variables, and set out in a general form the problem of the change of variables in a triple integral. By using the dependence relation between the differentials of the old and new variables:

$$dx = A\,dp + B\,dq + C\,dr,$$
$$dy = D\,dp + E\,dq + F\,dr,$$
$$dz = G\,dp + H\,dq + I\,dr$$

($A, B, C, \ldots$ are given functions of p, q, r), Lagrange tried to find a simple expression for the old element of volume $dxdydz$ in terms of the new one $dpdqdr$ and arrived—by means of some unconvincing reasoning—at the equation

$$dx\,dy\,dz = \begin{vmatrix} A & B & C \\ D & E & F \\ G & H & I \end{vmatrix} dp\,dq\,dr\dagger.$$

† Lagrange did not use our usual notation for a determinant and wrote the determinant in expanded form.

which in general is not true [cf. Sec. 359]. We have already mentioned that the mistake in Lagrange's reasoning was indicated by Ostrogradski in 1838; Ostrogradski also understood the geometrical significance of the expression which is *substituted* instead of the old "element of volume" $dxdydz$, *but which is not equal to it*. For more detail about this see Sec. 384.

§ 4. The elementary theory of a field

388. Scalars and vectors. The applications of integral calculus to the problems of mathematical physics and mechanics are often more convenient to use in vector form. The reader will find it useful to become acquainted with some fundamental ideas in vector analysis, which lead to the vector interpretation of integral forms and the formulae of integral calculus connecting them.

We presuppose that the reader is already familiar with the scalar or scalar value, which is completely characterized by its numerical value (as, for instance, volume, mass, density, temperature), and with the concept of a vector or vector quantity, which to be defined completely also requires an indication as to its direction (a transposition, velocity, acceleration, force and so on). In speaking of a vector, we will, as usual, be representing it as a directed segment. Let us agree to denote vectors by letters with arrows above them, $\vec{A}, \vec{r}, \vec{v}, \ldots$; the same letters without arrows: $A, r, v, \ldots$ will denote the lengths of the vectors:

$$A = |\vec{A}|, \quad r = |\vec{r}|, \quad v = |\vec{v}|, \ldots,$$

and the letters with indices, $A_x, r_y, v_n, \ldots$, will indicate their projections, respectively, on the axes $x, y, n, \ldots$ The projections A_x, A_y, A_z of the vector $\vec{A}$ on the coordinate axes determine it completely both in length (numerical value) and in direction.

We assume also that the reader possesses an elementary knowledge of vector algebra. We will confine ourselves to recalling that what we call the *scalar product* of the vectors $\vec{A}$ and $\vec{B}$ is the scalar (number)

$$\vec{A} \cdot \vec{B} = AB \cos(\vec{A}, \vec{B}),$$

which can be expressed in terms of the projections on the axes in the form:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z. \tag{1}$$

We will take as our basis from now on counterclockwise rotation and, hence, a right system of coordinates [Sec. 362, Remark]. Then the *vector product* of the vectors $\vec{A}$ and $\vec{B}$ is the vector with length $AB \sin(\vec{A}, \vec{B})$, perpendicular to both the factors and directed towards that side on which the rotation from $\vec{A}$ to $\vec{B}$ (through an angle less than 180°) is in a counterclockwise direction; it is denoted by $\vec{A} \times \vec{B}$. The projections of the vector product on the axes will be

$$A_y B_z - A_z B_y, \quad A_z B_x - A_x B_z, \quad A_x B_y - A_y B_x, \tag{2}$$

if we take as our basis a right system of coordinates.

389. Scalar and vector fields. If with every point M of a definite space domain (which may encompass the whole space) there is associated some scalar or vector quantity, then we say that there is given a *field* of this quantity, respectively, a *scalar* or a *vector* field. In the present section we shall constantly need to discuss such fields.

As an example of a scalar field we can take the field of temperature or electric potential. If we define the position of a point M by means of its coordinates with respect to some arbitrarily chosen coordinate system $Oxyz$, then the problem of a field of scalar values U is equivalent to the simple problem of a numerical function $U(x, y, z)$. We will always presuppose that this function has continuous partial derivatives with respect to all the variables. If these derivatives do not vanish simultaneously, then the equation

$$U(x, y, z) = C \qquad (C = \text{const})$$

defines some surface (without singular points), along which the quantity U has a constant value; such a surface is called a *level surface*. The whole domain under consideration is filled by these surfaces, so that through each of its points passes one and only one level surface. It is clear that the level surfaces do not intersect each other.

As an example of a vector field we can take a field of force or a field of velocities; we have already met such fields. If we take as a basis some system of coordinates $Oxyz$, then the specification of the field of a vector quantity $\vec{A}$ can be realized by specifying its projections on the axes

$$A_x(x, y, z), \quad A_y(x, y, z) \quad A_z(x, y, z) \tag{3}$$

as functions of the coordinates of the point M with which the quantity $\vec{A}$ is connected. We will also presuppose that these functions have continuous derivatives. In the study of a vector field an important part is played by the *vector lines*; a vector line is a curve whose direction at each of its points M coincides with the direction of the vector $\vec{A}$ corresponding to this point. If we recall [Sec. 212] that the direction cosines of the tangent to a curve are proportional to the differentials dx, dy, dz, then we obtain the fact that the vector line is characterized by the equations

$$\frac{dx}{A_x} = \frac{dy}{A_y} = \frac{dz}{A_z}.$$

With the hypothesis that the vector $\vec{A}$ does not vanish, we can prove, by using the "existence theorem" from the theory of systems of differential equations, that the whole domain under consideration is filled by vector lines, where through each of its points there passes one and only one such line. The vector lines do not intersect one another.

Sometimes it is necessary to consider surfaces consisting of vector lines; they are called *vector surfaces*. A vector surface is characterized by the fact that

at each of its points M the corresponding vector $\vec{A}(M)$ lies on the plane which is tangent to the surface at this point (or by the fact that the projection A_n of the vector $\vec{A}$ on the normal $\vec{n}$ to the surface at all its points is equal to zero). If we take any line in the domain under consideration which is not a vector line and through each of its points we draw the vector line, then the geometrical position of these lines gives us a vector surface. In the case when the given "directrix" line is closed, we obtain a tube-shaped vector surface, which is called a *vector tube*.

390. A derivative in a given direction. Gradient. Let there be given a scalar field $U(M)$. In many problems the "rate of variation" of this function is of interest, or its "derivative" *in any given direction.* We will make this concept more precise.

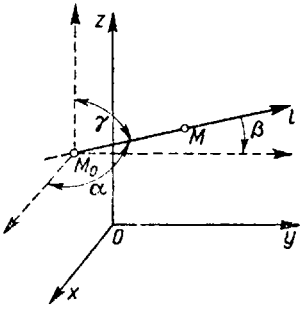

FIG. 68.

On an arbitrary directed straight line (or axis) l we take a constant point M_0 and a variable point M (Fig. 68); by M_0M we will mean the magnitude of the directed segment from M_0 to M, i.e. its length with the plus sign if the direction M_0M coincides with the direction of the axis l, and with the minus sign in the opposite case.

Let M approach M_0 without limit. *The derivative of the function $U(M)$ at the point M_0 in the direction of l* is what we call the limit

$$\lim_{M \to M_0} \frac{U(M) - U(M_0)}{M_0M};$$

it is denoted by the symbol:

$$\frac{\partial U(M_0)}{\partial l}. \tag{4}$$

This derivative characterizes the "rate of change" of the given function at the point M_0 in the direction of l.

We take as our basis some coordinate system $Oxyz$ and we suppose that the function of the coordinates

$$U(M) = U(x, y, z)$$

has continuous partial derivatives with respect to all its variables in the domain under consideration. Then, as we will prove, the derivative (4) exists and is expressed by the formula

$$\frac{\partial U}{\partial l} = \frac{\partial U}{\partial x}\cos\alpha + \frac{\partial U}{\partial y}\cos\beta + \frac{\partial U}{\partial z}\cos\gamma, \tag{5}$$

where all the derivatives are evaluated at the point $M_0(x_0, y_0, z_0)$, and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the axis l. If we set $M_0M = t$, then the coordinates of the variable point M of the axis can be expressed in the form:

$$x = x_0 + t\cos\alpha, \quad y = y_0 + t\cos\beta, \quad z = z_0 + t\cos\gamma.$$

The desired derivative will be the derivative of the compound function of t

$$U(x_0 + t\cos\alpha, \quad y_0 + t\cos\beta, \quad z_0 + t\cos\gamma)$$

when $t = 0$. By the rule for the differentiation of a compound function its value will be given exactly by formula (5).

We will introduce the vector $\vec{g}$, with the projections on the axes

$$\frac{\partial U}{\partial x}, \quad \frac{\partial U}{\partial y}, \quad \frac{\partial U}{\partial z}; \tag{6}$$

it is called the gradient *of the quantity U and is denoted by the symbol*

$$\vec{g} = \operatorname{grad} U.$$

If we return to formula (5) and denote by $\vec{\lambda}$ the ort (unit vector) drawn in the direction of l, then it can be rewritten thus:

$$\frac{\partial U}{\partial l} = \operatorname{grad} U \cdot \vec{\lambda} = \operatorname{grad}_l U.$$

Hence it is evident that this derivative attains its greatest value in the case when the direction of l coincides with the direction of the gradient and this greatest value is equal to

$$|\operatorname{grad} U| = \sqrt{\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2\right]}.$$

This enables us now, instead of the formal definition of the gradient given above, using the coordinate system, to state that: *the gradient of a scalar quantity U is the vector which in numerical value and in direction characterizes the greatest rate of change of the quantity U.* Here the coordinate system is not mentioned at all, which shows the actual independence of the concept of the gradient of the choice of this system.

It is easy to see that the direction of the gradient coincides with the direction of the normal to the level surface $U(x, y, z) = C$ passing through the given point.

Thus, the scalar field $U(M)$ generates a vector field of the gradient grad U

Hamilton† introduced the idea of the symbolic vector with "projections"

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z}$$

on the coordinate axes, which he called an "operator" and denoted by ∇. Using this notation we can write

$$\operatorname{grad} U = \nabla U.$$

In fact, if we multiply the given vector formally by the scalar U, then we obtain the vector with projections (6).

Examples. (1) Denoting by $\vec{r}$ the radius vector $\overrightarrow{OM}$ joining some constant point O to a variable point M in space, and by r its length, we set

$$U(M) = \varphi(r),$$

where φ is some scalar function of the positive scalar argument r, which has a derivative with constant sign. The level surfaces, clearly, will be spheres of radius r with centre at O, so that the direction of the gradient coincides with the radial axis, or the one exactly opposite to it, according to whether $\varphi'(r) > 0$ or < 0. It is easy to see that

$$\operatorname{grad}\varphi(r) = \varphi'(r)\frac{\vec{r}}{r}.$$

In particular,

$$\operatorname{grad}\frac{c}{r} = -\frac{c}{r^3}\vec{r} \qquad (c = \text{const}).$$

If we place a mass m at the point O and consider the field of Newtonian attraction, then its resultant $\vec{F}$ at the point M will be

$$\vec{F} = -\frac{m}{r^2}\frac{\vec{r}}{r} = -\frac{m}{r^3}\vec{r}$$

and, thus,

$$\vec{F} = \operatorname{grad}\frac{m}{r}.$$

The problem of whether a given vector field can be considered as the field of the gradient of some scalar quantity is of great importance. Essentially it requires nothing which is new to us; we will return to it below [Sec. 393].

(2) We will consider the field of the temperature U. Taking an element of the surface (dS) with a definite direction for the normal n, we consider the quantity dQ of heat flowing past this element in the direction of n in an infinitely small interval of time dt. The heat flows from the parts of the body or medium with greater heat to those with less heat, and moreover it flows the faster, the faster the temperature decreases. Usually we take it that the elementary quantity of heat dQ mentioned above is proportional to dS, dt and, finally, $|\partial U/\partial n|$. Denoting

† William Rowan Hamilton (1805–1865) was an English mathematician.

by $k>0$ the coefficient of proportionality ("the coefficient of internal heat conduction"), we can write

$$dQ = -k\,dS\,dt\,\frac{\partial U}{\partial n};$$

according to what was said above, the quantity of heat, dQ, is positive in the case when $\partial U/\partial n$ is negative, i.e. when the temperature U decreases in the direction of n.

If we introduce the so-called vector of the flow of heat

$$\vec{q} = -k\,\mathrm{grad}\,U,$$

then the expression for dQ can be rewritten in a shorter form:

$$dQ = dS\,dt\,q_n.$$

391. The flow of a vector through a surface. Now let there be given some vector field $\vec{A}(M)$, i.e. let the three functions (3) be given. We will take a surface (S) and, choosing a definite side on it, we denote by $\cos\lambda$, $\cos\mu$, $\cos\nu$ the direction cosines of the corresponding direction of the normal n. Then *the surface integral*

$$\iint\limits_{(S)} (A_x\cos\lambda + A_y\cos\mu + A_z\cos\nu)\,dS$$

is called the flow of the vector $\vec{A}$ *through the surface* (S) *to the given side*. If the same integral is written in the form

$$\iint\limits_{(S)} A_n\,dS,$$

then its independence of the choice of the coordinate system becomes clear.

We turn to some examples.

(1) The actual "flow" we have mentioned is connected with a certain problem in hydrostatics. We will consider the movement of a fluid in space; in the general case we do not presuppose that it is stationary, so that the velocity of the motion $\vec{v}$ depends not only on the position of the point M to which it refers, but also to the time t. We set ourselves the problem of evaluating the amount of liquid passing through the surface (S) to a definite side in an infinitely small interval of time dt. Through the element (dS) of the surface flows an amount of fluid which fills a cylinder with base dS and height $v_n dt$ (Fig. 69), where the normal n is presupposed directed to the chosen side†. If we denote by ϱ the density of the fluid, which may also depend both on the position of the point and on the time, then the mass of the fluid flowing through dS will be

$$\varrho\,dSv_n\,dt.$$

For the whole surface (S) we obtain

$$dt\iint\limits_{(S)} \varrho v_n\,dS.$$

† Cf. the footnote on p. 241 .

The amount of the fluid which flows in unit time is expressed by the integral

$$Q = \int\int_{(S)} \varrho v_n \, dS; \tag{7}$$

the reader will recognize this as the "flow of the vector" $\overrightarrow{\varrho v}$ through the surface (S).

Remark. Because we did not presuppose that the movement of the fluid was fixed, the quantity Q itself in fact, generally speaking, depends on the time and can be more precisely called the *velocity* of increase of the amount of fluid flowing through (S) in the moment of time under consideration.

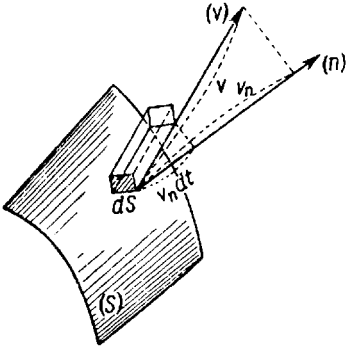

FIG. 69.

A similar remark can also be made about "the flow of an amount of heat through (S) in unit time" [see the following section]: all these are quantities having the nature of velocities.

(2) Analogously we can also speak of a flow of heat. It is easy to see that [with the notation of Sec. 390, *(2)*] in time dt an amount of heat equal to

$$dt \int\int_{(S)} q_n \, dS$$

flows through the surface (S).

If we refer to the amount of heat flowing in unit time, then we obtain

$$\int\int_{(S)} q_n \, dS,$$

i.e. "the flow of the vector" $\vec{q}$ through the surface (S). Hence the given vector

$$\vec{q} = -k \operatorname{grad} U$$

is "the vector of the flow of heat".

392. Ostrogradski's formula. Divergence. Turning to the general case of a vector field $\vec{A}$, we consider a body (V), bounded by a closed surface (S); we

will denote by n the outer normal to the surface. Then by Ostrogradski's formula [Sec. 380, (5)], if we set $P = A_x, Q = A_y, R = A_z$ in it, we can transform the flow of the vector $\vec{A}$ through the surface (S) to the outside into the triple integral:

$$\iint\limits_{(S)} A_n\, dS = \iint\limits_{(S)} (A_x \cos\lambda + A_y \cos\mu + A_z \cos\nu)\, dS$$

$$= \iiint\limits_{(V)} \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) dV.$$

The expression under the triple integral sign is called the divergence of the vector $\vec{A}$ and is denoted by the symbol

$$\operatorname{div} \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \tag{8}$$

Thus, Ostrogradski's formula can be rewritten in the form

$$\iint\limits_{(S)} A_n\, dS = \iiint\limits_{(V)} \operatorname{div} \vec{A}\, dV, \tag{9}$$

in which form it is most often applied.

The quantity which we have just introduced, divergence, is a scalar; but its definition is formally connected with the choice of the coordinate system. In order to be free of this defect, we proceed in the following way. We surround the point M with some body (V) with the surface (S) and we write down formula (9); if we divide both sides by the volume V of the body and take limits, by contracting the body (V) to the point M, then [Sec. 377, *(8)*] on the right we obtain precisely div $\vec{A}$ at the point M. Thus,

$$\operatorname{div} \vec{A} = \lim_{(V) \to M} \frac{\iint\limits_{(S)} A_n\, ds}{V}; \tag{10}$$

this equation may also be used as the definition of divergence, where in this form the definition does not depend on the choice of the coordinate system.

This time the vector field $\vec{A}$ generates a scalar field of the divergence div $\vec{A}$.

We note that the definition (8) of divergence can, with the help of Hamilton's symbolic vector, be rewritten thus:

$$\operatorname{div} \vec{A} = \nabla \vec{A};$$

this becomes clear if we recall that the expression (1) is the scalar product of two vectors.

Example. We will consider the motion of an incompressible fluid ($\varrho = 1$) when there are sources or outlets. The *realization of sources* contained inside a closed surface (S) is the quantity of fluid flowing through (S) in unit time, i.e. the flow of the vector velocity $\vec{v}$

$$\iint\limits_{(S)} v_n\, dS$$

[see Sec. 391, *(1)*]. If the sources are distributed continuously over the domain under consideration, then we introduce the concept of the *density of sources*. This is what we call the limit value of the realization of the sources in the body (V) surrounding the point M, considered to be of unit volume, i.e.

$$\lim_{(V)\to M} \frac{\iint\limits_{(S)} v_n\,dS}{V}.$$

But, as we have only just seen [see (10)], this limit is equal to $\operatorname{div}\vec{v}$; thus, $\operatorname{div}\vec{v}$ is the density of the sources.

Analogous reasoning can be carried out for heat flow when there are sources of heat, except that instead of the vector velocity we would have to take the vector of the flow of heat.

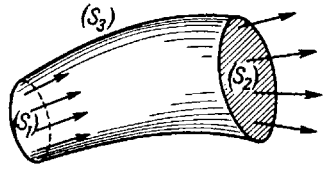

FIG. 70.

If $\operatorname{div}\vec{A} = 0$, the field is said to be *solenoidal*, i.e. *tubular*, from the Greek word $\sigma o\lambda\acute{\eta}\nu$ = tube. This condition—with the hypothesis that the part of space which the field occupies is simply connected (space-wise)—is equivalent to the requirement that *the flow of the vector* $\vec{A}$ *through an arbitrary closed surface bounding a body to the outside is equal to zero*:

$$\iint\limits_{(S)} A_n\,dS = 0$$

[Sec. 381, (B)]. Now we consider the vector tube (Fig. 70) between two arbitrary sections (S_1) and (S_2) of itself; the actual surface of the tube we denote by (S_3). Then, by what we have said,

$$\left\{\iint\limits_{(S_1)} + \iint\limits_{(S_2)} + \iint\limits_{(S_3)}\right\} A_n\,dS = 0,$$

where the normal in all cases is directed to the outside. Along the surface (S_3), clearly, $A_n = 0$ [Sec. 389]; if the direction of the normal to the section (S_1) varies in such a way that it coincides with the direction of the normal to (S_2), then we arrive at the equation

$$\iint\limits_{(S_1)} A_n\,dS = \iint\limits_{(S_2)} A_n\,dS.$$

Thus we obtain a remarkable property of a solenoidal field: *the flow of a solenoidal vector through a transverse section of a vector tube has a constant value*; it is called the intensity of the vector tube.

If we return to the hydromechanical interpretation of a vector field mentioned above, then it turns out that, in the case of an incompressible fluid when there are no sources (div $\vec{A} = 0$), the current of the fluid through a transverse section of a vector tube has the same value for all sections.

393. The circulation of a vector. Stokes's formula. Vortex. Again let there be given some vector field $\vec{A}(M)$. *The integral*

$$\int_{(l)} A_x\,dx + A_y\,dy + A_z\,dz = \int_{(l)} A_l\,dl,$$

taken along some curve (l) *within the limits of the domain under consideration is called the linear integral of the vector* $\vec{A}$ *along the curve* (l). In the case of a closed curve this integral is called the *circulation of the vector* $\vec{A}$ *along* (l).

If the field $\vec{A}$ is a field of force, then the linear integral expresses the work of the field of force when a point moves along the curve (l) [as in the plane case, cf. Sec. 335, *(1)*].

We will take some surface (S) bounded by a closed contour (l). Then by Stokes's formula, which is already known to the reader [Sec. 373, (11)], the circulation of the vector $\vec{A}$ along this contour can be expressed as a surface integral:

$$\int_{(l)} A_l\,dl = \iint_{(S)} \left\{\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\cos\lambda + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\cos\mu + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\cos\nu\right\} dS.$$

The vector with the projections on the axes

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z},\quad \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x},\quad \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \tag{11}$$

is called the vortex or rotor of the vector $\vec{A}$ and is denoted by the symbol

$$\operatorname{rot}\vec{A}\dagger.$$

Thus, in vector form Stokes's formula can be rewritten thus:

$$\int_{l} A_l\,dl = \iint_{(S)} \operatorname{rot}_n \vec{A}\,dS. \tag{12}$$

The circulation of a vector along a closed contour is equal to the flow of the vortex through the surface bounded by this contour. Here the direction of the circuit of the contour and the side of the surface must be connected, as is explained in Sec. 362.

† From the English word "rotation"; the notation curl $\vec{A}$ is also used, from the English word "curl".

The definition of the concept of a vortex given above has the usual defect: in it a definite coordinate system is used. Taking any direction n starting from the given point M, we surround it by an area (σ) with the contour (λ) on the plane perpendicular to n (Fig. 71). Then by Stokes's formula

$$\int_{(\lambda)} A_\lambda d\lambda = \iint_{(\sigma)} \operatorname{rot}_n \vec{A}\, d\sigma;$$

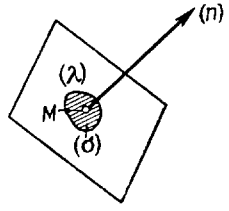

FIG. 71.

dividing both sides by the area σ of the given part of the surface and "compressing" the latter to the given point, we arrive at the limit

$$\operatorname{rot}_n \vec{A} = \lim_{(\sigma)\to M} \frac{\int_{(\lambda)} A_\lambda d\lambda}{\sigma}\dagger.$$

In this way we succeed in defining the projection of the vector rot $\vec{A}$ on any axis, and this means the vector itself is defined without reference to a previously chosen coordinate system.

We emphasize that here the vector field $\vec{A}$ generates another vector field of the vortex rot $\vec{A}$.

By means of the Hamiltonian vector ∇ we can simply rewrite the definition of the vortex thus:

$$\operatorname{rot} \vec{A} = \nabla \times \vec{A}$$

[see the expression (2) for the projections of a vector product].

Example. We will consider the arbitrary motion of some rigid body around a fixed point O (Fig. 72). As is proved in kinematics, for any moment of time the field of the velocity $\vec{v}$ of the points of the body is defined by the formula

$$\vec{v} = \vec{\omega} \times \vec{r},$$

where $\vec{\omega}$ is the instantaneous "angular velocity", and $\vec{r}$ is the radius vector joining the point O to an arbitrary point M of the body. The projections of this vector on the axes of an arbitrary system $Oxyz$ will be [see (2)]

$$\omega_y z - \omega_z y, \quad \omega_z x - \omega_x z, \quad \omega_x y - \omega_y x.$$

† It is easy to see here the original differentiation with respect to a domain; we leave it to the reader to justify this.

If, using the expressions in (11), we consider the projections of the vortex for this field, then we obtain $2\omega_x$, $2\omega_y$, $2\omega_z$, so that

$$\vec{\omega} = \tfrac{1}{2} \operatorname{rot} \vec{v}.$$

Thus, to within a numerical factor, the rotor of the field of velocity $\vec{v}$ gives precisely the instantaneous angular velocity of the rotation; this is the reason for the term "rotor".

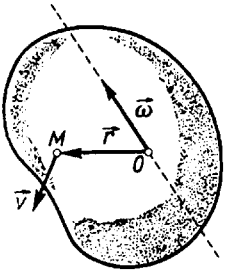

FIG. 72.

Now we turn to the question of under what conditions a given vector field $\vec{A}$ will be the field of the gradient for some scalar quantity U:

$$\vec{A} = \operatorname{grad} U. \tag{13}$$

This equation is equivalent to the following three:

$$A_x = \frac{\partial U}{\partial x}, \quad A_y = \frac{\partial U}{\partial y}, \quad A_z = \frac{\partial U}{\partial z},$$

i.e. it is equivalent to the assertion that the expression

$$A_x\,dx + A_y\,dy + A_z\,dz$$

is the (complete) differential of the function $U(x, y, z)$. When equation (13) holds, the field $\vec{A}$ is called *potential*, and the transforming function U is called the potential function of the field.

If we limit ourselves by the hypothesis that the domain under consideration is (superficially) simply connected, then, rephrasing the facts which we already know [Sec. 374, see condition (B)], we can say that *for the field $\vec{A}$ to be potential, it is necessary and sufficient that in the whole domain the equations*

$$\frac{\partial A_z}{\partial y} = \frac{\partial A_y}{\partial z}, \quad \frac{\partial A_x}{\partial z} = \frac{\partial A_z}{\partial x}, \quad \frac{\partial A_y}{\partial x} = \frac{\partial A_x}{\partial y}$$

should be satisfied, i.e. that rot $\vec{A}$ should vanish. The concept of a potential field coincides with the concept of an "irrotational" field. In such a field the circulation

along a closed contour will be zero; if we take the linear integral along a curve joining any two points, its value does not depend on the form of the curve.

All these facts have a natural interpretation in terms of "work" for the case of a potential field of force. An example of this, as we know, will be the field of Newtonian attraction both in the case of separate centres of attraction, and also when there is a continuous distribution of attracting mass.

§ 5. Multiple integrals

394. The volume of an m-dimensional body and the m-tuple integral. The requirements of analysis and its applications are not exhausted by the types of definite integral already studied: single, double and triple.

As in the definition of simple, double and triple integrals, we used the concepts of the length of a segment, the area of a plane figure, and the volume of a space body, so the basis of the definition of an m-tuple integral is the concept of the volume† of an m-dimensional domain. For the simplest m-dimensional domain—the m-dimensional rectangular parallelepiped

$$[a_1, b_1; a_2, b_2; \ldots; a_m, b_m] \tag{1}$$

—the volume is the product of the lengths of the sides

$$(b_1 - a_1)(b_2 - a_2) \cdot \ldots \cdot (b_m - a_m).$$

It is self-evident what is meant by the volume of a body consisting of a finite number of such parallelepipeds. It can be shown in a quite elementary way that the volume does not depend on the way in which the body is divided up into parallelepipeds.

By considering such "parallelepipedal" bodies, contained in a given m-dimensional body (V) and surrounding it, we can construct the concept of the volume V for the body (V) in the usual way [cf. Sec. 197]. We will discuss only bodies for which the volumes exist; they are known to exist for bodies bounded by smooth or piece-wise smooth surfaces‡, in particular, for the simplest m-dimensional domains: the m-dimensional pyramids

$$x_1 \geqslant 0, \quad x_2 \geqslant 0, \ldots, \quad x_m \geqslant 0, \quad x_1 + x_2 + \ldots + x_m \leqslant h$$

and the m-dimensional spheres

$$x_1^2 + x_2^2 + \ldots + x_m^2 \leqslant r^2;$$

we will calculate their volumes below.

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† We have decided to keep this term, although its meaning, of course, varies with m: we speak about an "m-dimensional" volume.

‡ A smooth surface here means a form in m-dimensional space, defined by m parametric equations with $m-1$ parameters, where the functions of the parameters occurring in the equations must be continuous together with their partial derivatives, and the determinants of the $(m-1)$th order of the matrix of the derivatives must not vanish simultaneously.

In the domain (V) let there be given a function of the m variables $f(x_1, x_2, \ldots, x_m)$; then, dividing up this domain into elementary parts and repeating the other operations which are already so familiar to us [cf. Sec. 376], we arrive at the concept of an m-tuple integral

$$I = \overbrace{\int \ldots \int}^{m}_{(V)} f(x_1, x_2, \ldots, x_m)\, dx_1, dx_2 \ldots dx_m. \tag{2}$$

In the case when the function under the integral is continuous it does in fact exist.

The evaluation of such an integral leads to the evaluation of integrals of lower order, up to simple integrals. In the case when the domain of integration (V) is a rectangular parallelepiped (1), a formula analogous to formula (6), Sec. 378, holds:

$$I = \int_{a_1}^{b_1} dx_1 \int_{a_2}^{b_2} dx_2 \ldots \int_{a_m}^{b_m} f(x_1, x_2, \ldots, x_m)\, dx_m. \tag{3}$$

For a domain of more general form, characterized by the inequalities

$$x_1^0 \leqslant x_1 \leqslant X_1,\ x_2^0(x_1) \leqslant x_2 \leqslant X_2(x_1), \ldots,$$

$$x_m^0(x_1, \ldots, x_{m-1}) \leqslant x_m \leqslant X_m(x_1, \ldots, x_{m-1}),$$

a formula analogous to formula (6a), Sec. 378 is applicable:

$$I = \int_{x_1^0}^{X_1} dx_1 \int_{x_2^0(x_1)}^{X_2(x_1)} dx_2 \ldots \int_{x_m^0(x_1, \ldots, x_{m-1})}^{X_m(x_1, \ldots, x_{m-1})} f(x_1, x_2, \ldots, x_m)\, dx_m. \tag{4}$$

In a similar way (for the corresponding type of domain, which in each case is not difficult to establish) other formulae analogous to the formulae (5a) and (7a), Sec. 378 hold, where the evaluation of the m-tuple integral reduces to the successive evaluation of integrals of lower orders, the sum of which is m.

All this is proved in exactly the same way as for the case $m = 2$ or $m = 3$, without the introduction of any new ideas, so that there is no need to linger over the proof.

Remark. Ostrogradski, in a memoir in 1834, first explained with care the limits of integration with respect to the separate variables $x_1, x_2, \ldots, x_m$, to which the evaluation of a multiple integral is reduced, where it is extended over all the values of these variables satisfying an inequality of the form $L(x_1, x_2, \ldots, x_m) \leqslant 0$. Here we find the formula generalizing Ostrogradski's formula which we know [Sec. 380, (4)] to the case of any number of variables and connecting an integral taken over an $(m-1)$-dimensional closed surface with some m-dimensional integral extended over the body bounded by it.

395. Examples. *(1)* To find the volume T_m of the m-dimensional pyramid

$$(T_m) : x_1 \geqslant 0, \ \ldots, \ x_m \geqslant 0, \quad x_1 + \ldots + x_m \leqslant h.$$

Solution. We have

$$T_m = \overbrace{\int \dots \int}^{m}_{(T_m)} dx_1, dx_2, \dots, dx_m = \int_0^h dx_1 \int_0^{h-x_1} dx_2 \dots \int_0^{h-x_1-\dots-x_{m-1}} dx_m.$$

By replacing the variables in these simple integrals successively by the formulae

$$x_1 = h\xi_1,\ x_2 = h\xi_2,\ \dots,\ x_m = h\xi_m,$$

we arrive at the result

$$T_m = h^m \int_0^1 d\xi_1 \int_0^{1-\xi_1} d\xi_2 \dots \int_0^{1-\xi_1-\dots-\xi_{n-1}} d\xi_n$$

$$= h^n \overbrace{\int \dots \int}^{m}_{\substack{\xi_1 \geqslant 0, \dots, \xi_n \geqslant 0 \\ \xi_1 + \dots + \xi_n \leqslant 1}} d\xi_1 \dots d\xi_n = \alpha_n h^n,$$

if we denote by α_n the value of the integral, similar to the one suggested, but corresponding to $h = 1$.

On the other hand, we have (using the result obtained)

$$\alpha_n = \int_0^1 d\xi_n \overbrace{\int \dots \int}^{n-1}_{\substack{\xi_1 \geqslant 0, \dots, \xi_{n-1} \geqslant 0 \\ \xi_1 + \dots + \xi_{n-1} \leqslant 1 - \xi_n}} d\xi_1 \dots d\xi_{n-1} = \alpha_{n-1} \int_0^1 (1-\xi_n)^{n-1} d\xi_n = \frac{\alpha_{n-1}}{n}.$$

The recurrence relation which we have found (taking into account that $\alpha_1 = 1$) gives us

$$\alpha_n = 1/n!,$$

so that finally

$$T_n = h^n/n!.$$

(2) To find the volume V_m of the m-dimensional sphere [Sec. 126]

$$(V_m): \quad x_1^2 + x_2^2 + \dots + x_m^2 \leqslant R^2.$$

Solution. In this case we are discussing the evaluation of the integral

$$V_m = \overbrace{\int \dots \int}^{m}_{x_1^2 + \dots + x_m^2 \leqslant R_2} dx_1, dx_2, \dots, dx_m.$$

Letting

$$x_1 = R\xi_1, \quad x_2 = R\xi_2, \dots, x_m = R\xi_m,$$

as we have just done, it is easy to obtain that $V_m = \beta_m R_*^m$, where the numerical coefficient β_m expresses the volume of the m-dimensional sphere of radius 1.

To determine β_m we make the transformation

$$\beta_m = \overbrace{\int \dots \int}^{m} \limits_{\xi_1^2 + \xi_2^2 + \dots + \xi_m^2 \leqslant 1} d\xi_1, d\xi_2, \dots, d\xi_m$$

$$= \int_{-1}^{1} d\xi m \overbrace{\int \dots \int}^{m-1} \limits_{\xi_1^2 + \dots + \xi_{m-1}^2 \leqslant 1 - \xi_m^2} d\xi_1 \dots d\xi_{m-1}.$$

The inner integral is the volume of the $(m-1)$-dimensional sphere of radius $\sqrt{(1-\xi_m^2)}$ and, hence, is equal to $\beta_{m-1}(1-\xi_m^2)^{(m-1)/2}$. Substituting, we again arrive at the recurrence relation

$$\beta_m = 2\beta_{m-1} \int_0^{\pi/2} \sin^m \theta \, d\theta$$

or [see Sect. 312, *(2)*]

$$\beta_m = \beta_{m-1} \sqrt{\pi} \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}.$$

Because $\beta_1 = 2$, then an easy calculation gives

$$\beta_m = \frac{\pi^{m/2}}{\Gamma\left(\frac{m}{2}+1\right)}.$$

The desired volume is equal to

$$V_m = \frac{\pi^{m/2}}{\Gamma\left(\frac{m}{2}+1\right)} R^m.$$

For the cases of m, even or odd, we obtain the formulae

$$V_{2n} = \frac{\pi^n}{n!} R^{2n}, \qquad V_{2n+1} = \frac{2(2\pi)^n}{(2n+1)!!} R^{2n+1}.$$

In particular, for V_1, V_2, V_3, naturally, we find the well-known values $2R$, πR^2, $(4/3)\pi R^3$.

CHAPTER 24

FOURIER SERIES

§ 1. Introduction

396. Periodic values and harmonic analysis. In science and technology it is often necessary to discuss periodic phenomena, i.e. such as are found to recur in their previous form after a definite interval of time T, called the period. An example is the set motion of a steam engine, which after a complete revolution again passes through its initial position; also there is the phenomenon of an alternating current, and so on. The different values connected with the periodic phenomena under consideration return to their original values after a period T and are, consequently, periodic functions of the time t, characterized by the equation

$$\varphi(t+T) = \varphi(t).$$

Such, for instance, are the force and intensity of an alternating current or—in the example of the steam engine—the path, velocity and acceleration of the cross-head, the pressure of the steam, the tactile strength in the teeth of the crankwheel, and so on.

The simplest of the periodic functions (if we do not consider constant functions) is the *sine function*: $A\sin(\omega t+\alpha)$, where ω is the frequency, connected to the period T by the relation:

$$\omega = \frac{2\pi}{T}. \tag{1}$$

From similar simple periodic functions can be constructed more complicated ones. It is clear in advance that the component sine functions must be of different frequencies, because, as it is easy to show, the addition of sine functions of the same frequency provides nothing new, because it reduces to a sine function which has

the same frequency. However, if we add several functions of the form

$$\left.\begin{aligned} y_0 = A_0, \quad y_1 = A_1 \sin(\omega t + \alpha_1), \quad y_2 = A_2 \sin(2\omega t + \alpha_2), \\ y_3 = A_3 \sin(3\omega t + \alpha_3), \ldots \end{aligned}\right\} \tag{2}$$

which, if we do not assume them to be constant, have the frequencies

$$\omega, 2\omega, 3\omega, \ldots,$$

with the least common multiple ω, and the periods

$$T, \tfrac{1}{2}T, \tfrac{1}{3}T, \ldots,$$

then we obtain a periodic function (with period T), but one which is essentially distinct from the functions of the type (2).

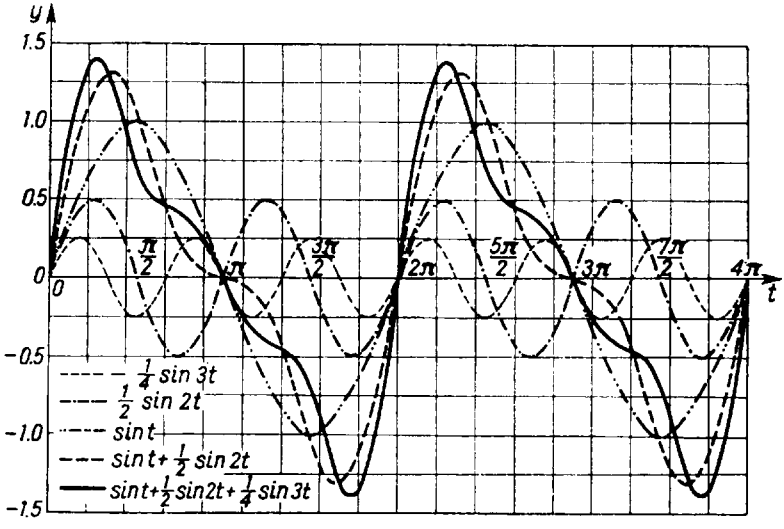

FIG. 73.

As an example we again add together here (Fig. 73) the three sine functions:

$$\sin t + \tfrac{1}{2}\sin 2t + \tfrac{1}{4}\sin 3t;$$

the graph of this function differs considerably in its character from that of a sine function. This is even more true for the sum of an infinite series consisting of functions of the form (2).

Now it is natural to pose the converse question: can a given periodic function $\varphi(t)$ of period T be represented in the form of the sum of a finite or even infinite set of sine functions of the form (2)? As we shall see below, for a very wide class of functions we can give the answer to this question in the affirmative, but only if we take into account an infinite sequence of functions (2). For functions of this class the expansion in "trigonometrical series" holds:

$$\varphi(t) = A_0 + A_1 \sin(\omega t + \alpha_1) + A_2 \sin(2\omega t + \alpha_2)$$

$$+ A_3 \sin(3\omega t + \alpha_3) + \ldots = A_0 + \sum_{n=1}^{\infty} A_n \sin(n\omega t + \alpha_n), \quad (3)$$

where $A_0, A_1, \alpha_1, A_2, \alpha_2, \ldots$ are constants having special values for each such function, and the frequency ω is given by the formula (1).

Geometrically this means that *the graph of a periodic function is obtained by superimposing a series of sine graphs.* If, however, we interpret each sine function mechanically as representing harmonic motion, then we can also say that here *the compound oscillation, characterized by the function $\varphi(t)$, can be factorized into separate harmonic oscillations.* In connection with this the separate sine functions, which are part of the factorization (3), are called the *harmonic components* of the function $\varphi(t)$ or simply its *harmonics* (the first, second and so on). This process of decomposing a periodic function into its harmonics is called *harmonic analysis.*

If we take as the independent variable

$$x = \omega t = \frac{2\pi t}{T},$$

then we obtain a function of x:

$$f(x) = \varphi\left(\frac{x}{\omega}\right),$$

which is also periodic, but with the standard period 2π. The expansion (3) takes the form

$$f(x) = A_0 + A_1 \sin(x + \alpha_1) + A_2 \sin(2x + \alpha_2)$$

$$+ A_3 \sin(3x + \alpha_3) + \ldots = A_0 + \sum_{n=1}^{\infty} A_n \sin(nx + \alpha_n). \quad (4)$$

Expanding the terms of this series by the formula for the sine of a sum and setting

$$A_0 = a_0, \quad A_n \sin\alpha_n = a_n, \quad A_n \cos\alpha_n = b_n \quad (n = 1, 2, 3, \ldots),$$

we arrive at the final form of the trigonometrical expansion:

$$f(x) = a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + (a_3 \cos 3x + b_3 \sin 3x) + \ldots$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx); \tag{5}$$

this is the form we will always use†. Here the function of the angle x, having period 2π, is an expansion in cosines and sines of angles which are multiples of x.

We arrived at the expansion of the function in a trigonometrical series by starting from periodic, oscillating phenomena and the functions connected with them. *It is important, however, to note at once that similar expansions are often also useful in the investigation of functions given only in a definite finite interval and in no way generated by oscillating phenomena.*

397. The determination of coefficients by the Euler–Fourier method. In order to establish the possibility of a trigonometrical expansion (5) for a given function $f(x)$, having period 2π, it is necessary to start from a definite set of coefficients $a_0, a_1, b_1, a_2, b_2, \ldots$ We will indicate a way of defining them, which was applied by Euler in the second half of the eighteenth century, and independently at the beginning of the nineteenth century by Fourier.

From now on we will presuppose that the function $f(x)$ is continuous or piece-wise continuous in the interval $[-\pi, \pi]$‡.

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† From this expansion, of course, it is easy, when necessary, to obtain, conversely, the expansion of the form (4).

‡ A function $f(x)$ is said to be piece-wise continuous in the interval $[a, b]$ if it is continuous in it, with the exception of a finite number of points, where there are jumps. Thus, the interval $[a, b]$ can be decomposed into a finite number of part intervals, in each of which separately the function $f(x)$ is uninterruptedly continuous, even including the end-points, if there we replace the values of the function by its limit values. A piece-wise continuous function may be "put together" in any way from several continuous functions, provided that only at the "junction points" (and equally at the end-points of the interval) its values are shown to be singular.

Let us assume that the expansion (5) exists, and we will integrate it termwise from $-\pi$ to π; we obtain

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\pi a_0 + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos nx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\,dx\right].$$

But, as it is easy to see,

$$\left.\begin{aligned}\int_{-\pi}^{\pi}\cos nx\,dx &= \frac{\sin nx}{n}\Big|_{-\pi}^{\pi} = 0\\ \text{and}\qquad\qquad &\qquad\qquad (6)\\ \int_{-\pi}^{\pi}\sin nx\,dx &= -\frac{\cos nx}{n}\Big|_{-\pi}^{\pi} = 0.\end{aligned}\right\}$$

Therefore all the terms after the summation sign will be zero, and we find, finally,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx. \tag{7}$$

In order to find the value of the coefficient a_m, we multiply both sides of equation (5), which all the time we presuppose to be satisfied, by $\cos mx$ and we again integrate termwise in the same interval:

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = a_0\int_{-\pi}^{\pi}\cos mx\,dx$$

$$+\sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos nx\cos mx\,dx + b_n\int_{-\pi}^{\pi}\sin nx\cos mx\,dx\right].$$

The first term on the right vanishes in view of (6). Further we have

$$\int_{-\pi}^{\pi}\sin nx\cos mx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}[\sin(n+m)x + \sin(n-m)x]\,dx = 0, \tag{8}$$

$$\int_{-\pi}^{\pi}\cos nx\cos mx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}[\cos(n+m)x + \cos(n-m)x]\,dx = 0, \tag{9}$$

if $n \neq m$, and, finally,

$$\int_{-\pi}^{\pi} \cos^2 mx\, dx = \int_{-\pi}^{\pi} \frac{1+\cos 2mx}{2}\, dx = \pi. \tag{10}$$

Thus all the integrals after the summation sign vanish, except the integral in which the coefficient a_m occurs as a factor. Hence this coefficient is also determined:

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx\, dx \qquad (m = 1, 2, 3, \ldots). \tag{11}$$

Analogously, by multiplying the expansion (5) by $\sin mx$ and then integrating termwise, we determine the coefficient for the sine:

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx\, dx \qquad (m = 1, 2, 3, \ldots). \tag{12}$$

Here, besides (6) and (8), we also use the easily verified relation:

$$\int_{-\pi}^{\pi} \sin nx \sin mx\, dx = 0, \tag{13}$$

if $n \neq m$, and

$$\int_{-\pi}^{\pi} \sin^2 mx\, dx = \pi. \tag{14}$$

The formulae (7), (11) and (12) are known as *the Euler–Fourier formulae*; the coefficients evaluated by means of these formulae are called the *Fourier coefficients* of the given function, and the trigonometrical series formed with their help is called its *Fourier series.* We will be concerned exclusively with Fourier series in the present chapter.

Now we will explain the logical significance of the arguments given. Because we started from the hypothesis that the trigonometrical expansion (5) holds, the question of whether it corresponds to reality naturally remains open. But are the considerations convincing, by which, following Euler and Fourier, we defined the coefficients of the expansion (5), even with the hypothesis that it exists? We repeatedly used termwise integration of a series, and this operation is not always permissible [Sect. 269]. A sufficient condition for

its applicability is the uniform convergence of the series. Therefore to establish this rigorously we can only assume the following:

If the function $f(x)$ can be expanded into a uniformly convergent trigonometrical series (5)†, *then the latter will necessarily be its Fourier series.*

If, however, we do not presuppose the uniform convergence in advance, then our considerations do not even prove that the function can be expanded in a Fourier series. What is the significance of the considerations mentioned? They can only be regarded as a guide, so that in the search for a trigonometrical expansion of the given function we begin with its Fourier series, being obliged (with complete rigorousness) to establish the conditions under which it converges, and, moreover, precisely to the given function.

Until this is done we are only entitled to consider formally the Fourier series of the given function $f(x)$, but we can assert nothing about it, except that it is "generated" by the function $f(x)$. Its connection with the function f is usually denoted thus:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{5a}$$

which avoids the equality sign.

398. Orthogonal systems of functions. The argument in the previous section is an example of the kind of reasoning which it is often necessary to use in mathematical analysis in the study of many expansions.

We will call the two functions $\varphi(x)$ and $\psi(x)$, defined in the interval $[a, b]$, orthogonal in this interval, if their product has integral equal to zero:

$$\int_a^b \varphi(x)\psi(x)\,dx = 0.$$

We will consider a system of functions $\{\varphi_n(x)\}$, defined in the interval $[a, b]$ and continuous in it or, at least, piece-wise continuous. If the functions of the given system are orthogonal by pairs:

$$\int_a^b \varphi_n(x)\varphi_m(x)\,dx = 0, \qquad (n, m = 0, 1, 2, \ldots;\ n \neq m) \tag{15}$$

† We note that uniform convergence is preserved when all the terms of the series are multiplied by the bounded functions $\cos mx$, $\sin mx$.

then it is called an *orthogonal system* of functions. Here we will always presuppose that

$$\int_a^b \varphi_n^2(x)dx = \lambda_n > 0. \tag{16}$$

When the condition $\lambda_n = 1$ $(n = 0, 1, 2, \ldots)$ is satisfied, the system is called *normal*. If, however, these conditions are not satisfied, then if we desire we can go over to the system $\{\varphi_n(x)/\sqrt{\lambda_n}\}$, which will clearly be normal.

An important example of an orthogonal system of functions is precisely the trigonometrical system

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx, \ldots \tag{17}$$

in the interval $[-\pi, \pi]$, which we considered above; its orthogonality follows from the relations (6), (8), (9) and (13). However, it will not be normal in view of (10) and (14). By multiplying the trigonometrical functions (17) by the proper factors, it is easy to obtain the normal system:

$$\frac{1}{\sqrt{(2\pi)}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \ldots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}}, \ldots. \tag{17*}$$

In the interval $[a, b]$ let there be given any orthogonal system $\{\varphi_n(x)\}$. We seek to expand a function $f(x)$ defined in the interval $[a, b]$ in a "series with respect to the functions φ" in the form

$$f(x) = c_0\varphi_0(x) + c_1\varphi_1(x) + \ldots + c_n\varphi_n(x) + \ldots. \tag{18}$$

To determine the coefficients of this expansion, assuming that it is possible, we proceed in the same way as we did in the special case above. Thus, multiplying both sides of the expansion by $\varphi_m(x)$, we integrate it termwise:

$$\int_a^b f(x)\varphi_m(x)dx = \sum_{n=0}^{\infty} c_n \int_a^b \varphi_n(x)\varphi_m(x)dx.$$

By virtue of the orthogonality [see (15) and (16)], all the integrals on the right, except one, will be zero, and it is easy to obtain:

$$c_m = \frac{1}{\lambda_m}\int_a^b f(x)\varphi_m(x)dx \qquad (m = 0, 1, 2, \ldots) \tag{19}$$

[formulae (7), (11), (12) are special cases of this formula].

The series (18) with coefficients found from formula (19) is called the *(generalized) Fourier series* of the given function, and its coefficients are called its *(generalized) Fourier coefficients* with respect to the system $\{\varphi_n(x)\}$. Formula (19) turns out to be particularly simple in the case of a normal system; then

$$c_m = \int_a^b f(x)\varphi_m(x)\,dx. \tag{19a}$$

Of course, we can repeat here all the remarks which we included in the preceding section. The generalized Fourier series constructed for the given function $f(x)$ is connected with it only formally. In the general case also the connection between the function $f(x)$ and its (generalized) Fourier series is denoted by:

$$f(x) \sim \sum_0^\infty c_n\varphi_n(x). \tag{18a}$$

The convergence of this series to the function $f(x)$, as in the case of the trigonometrical series, is still subject to investigation.

§ 2. The expansion of functions in Fourier series

399. Statement of the problem. Dirichlet's integral. Let $f(x)$ be a continuous or piece-wise continuous function with period 2π. We will calculate the constants (its Fourier coefficients):

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\cos mu\,du, \quad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\sin mu\,du \tag{1}$$

$$(m = 0, 1, 2, \ldots) \qquad (m = 1, 2, 3, \ldots)$$

and from them construct the Fourier series for our function

$$f(x) \sim \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m\cos mx + b_m\sin mx). \tag{2}$$

The reader will notice here a slight difference from the notation of Sec. 397: the coefficient a_0 we now determine by the general formula for a_m when $m = 0$, as opposed to formula (7) of the section mentioned, but on the other hand we write the free term of the series in the form $a_0/2$.

Remark. If the function $F(u)$ is piece-wise continuous in any finite interval and has period 2π in it, so that

$$F(u+2\pi) = F(u),$$

then *the value of the integral*

$$\int_{\alpha}^{\alpha+2\pi} F(u)\,du$$

in an interval of length 2π does not depend on α.

In fact, if we limit ourselves to the case of a continuous function F, we have

$$\int_{\alpha}^{\alpha+2\pi} F(u)\,du = \int_{\alpha}^{0} F(u)\,du + \int_{0}^{2\pi} F(u)\,du + \int_{2\pi}^{\alpha+2\pi} F(u)\,du.$$

If in the last integral we make the substitution $u = 2\pi + t$, then it reduces to the integral

$$\int_{0}^{\alpha} F(t+2\pi)\,dt = \int_{0}^{\alpha} F(t)\,dt$$

and it will differ from the first integral only in sign. Thus the integral under consideration is equal to the integral

$$\int_{0}^{2\pi} F(u)\,du,$$

which does not contain α. It is easy to extend this result to the case of any piece-wise continuous function.

We will use this remark in what follows. In particular, in the formulae (1) defining the Fourier coefficients the integrals can be taken over any interval of length 2π; for instance, we can write

$$a_m = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos mx\,dx, \qquad b_m = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin mx\,dx \qquad \text{(1a)}$$

$$(m = 0, 1, 2, \ldots) \qquad (m = 1, 2, 3, \ldots)$$

and so on.

In order to investigate the behaviour of the series (2) at any definite point $x = x_0$, we form the convenient expression for its partial sum

$$s_n(x_0) = \frac{a_0}{2} + \sum_{m=1}^{n} (a_m \cos mx_0 + b_m \sin mx_0).$$

We substitute instead of a_m and b_m their integral expressions (1) and we put the constant numbers $\cos mx_0$, $\sin mx_0$ under the integral signs:

$$\begin{aligned} s_n(x_0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u)\,du \\ &\quad + \sum_{m=1}^{n} \frac{1}{\pi} \int_{-\pi}^{\pi} f(u)\,[\cos mu \cos mx_0 + \sin mu \sin mx_0]\,du \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) \left\{ \frac{1}{2} + \sum_{m=1}^{n} \cos m(u - x_0) \right\} du. \end{aligned}$$

It is easy to verify the identity

$$\begin{aligned} &\frac{1}{2} + \sum_{m=1}^{n} \cos m\alpha \\ &\quad = \frac{1}{2\sin\frac{\alpha}{2}} \left\{ \sin\frac{\alpha}{2} + \sum_{m=1}^{n} \left[\sin\left(m + \frac{1}{2}\right)\alpha - \sin\left(m - \frac{1}{2}\right)\alpha \right] \right\} \\ &\quad = \frac{\sin(2n+1)\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}}. \end{aligned}$$

Using it for the transformation of the expression under the integral, we finally obtain

$$s_n(x_0) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) \frac{\sin(2n+1)\frac{u - x_0}{2}}{2\sin\frac{u - x_0}{2}}\,du. \tag{3}$$

This integral is usually called *Dirichlet's integral* (although Fourier discovered it considerably earlier).

Because we are here discussing functions of u with period 2π, the interval of integration $[-\pi, \pi]$ in the remark made above can be replaced, for instance, by the interval $[x_0 - \pi, x_0 + \pi]$:

$$s_n(x_0) = \frac{1}{\pi} \int_{x_0-\pi}^{x_0+\pi} f(u) \frac{\sin(2n+1)\frac{u-x_0}{2}}{2\sin\frac{u-x_0}{2}} du.$$

By the substitution $t = u - x_0$ we transform this integral to the form

$$s_n(x_0) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x_0 + t) \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t} dt.$$

Then, dividing the integral into two parts, $\int_0^{\pi} + \int_{-\pi}^{0}$, and reducing the second integral, by changing the sign of the variable, to the interval $[0, \pi]$, we arrive at the following final expression for the partial sum of a Fourier series:

$$s_n(x_0) = \frac{1}{\pi} \int_0^{\pi} [f(x_0 + t) + f(x_0 - t)] \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t} dt. \qquad (4)$$

Thus, the problem reduces to the investigation of the behaviour of this integral, containing the parameter n. *The origin of the problem represented here lies in the fact that in this case the passage to a limit under the integral sign cannot be used*†, which until now [see Chapter 18] has served us as a natural means for finding the limit of an integral containing a parameter. In this chapter we will have to deal with such a state of affairs systematically.

400. A fundamental lemma. Before we continue our investigations, we will prove the following assertion, important in what follows, which is due to Riemann:

† In the given case the expression under the integral has no limit at all as $n \to \infty$.

If the function $g(t)$ is continuous or piece-wise continuous in some finite interval $[a, b]$, then

$$\lim_{p\to\infty}\int_a^b g(t)\sin pt\,dt = 0$$

and, analogously,

$$\lim_{p\to\infty}\int_a^b g(t)\cos pt\,dt = 0.$$

It is sufficient to carry out the proof for the first of these limits, presupposing the function $g(t)$ continuous.

We note as a preliminary that, for any finite interval $[\alpha, \beta]$, we have the following approximation:

$$\left|\int_\alpha^\beta \sin pt\,dt\right| = \left|\frac{\cos p\alpha - \cos p\beta}{p}\right| \leqslant \frac{2}{p}. \tag{5}$$

We divide up the interval $[a, b]$ into n parts by the points

$$a = t_0 < t_1 < \ldots < t_i < t_{i+1} < \ldots < t_n = b \tag{6}$$

and corresponding to this we also divide up the integral

$$\int_a^b g(t)\sin pt\,dt = \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} g(t)\sin pt\,dt.$$

Denoting by m_i' the least of the values of $g(t)$ in the ith interval, we can transform this expression into

$$\int_a^b g(t)\sin pt\,dt = \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} [g(t) - m_i]\sin pt\,dt + \sum_{i=0}^{n-1} m_i \int_{t_i}^{t_{i+1}} \sin pt\,dt.$$

If ω_i is the oscillation of the function $g(t)$ in the ith interval, then within its limits $g(t) - m_i \leqslant \omega_i$; by taking into account the inequality (5) it is now easy to obtain the approximation for our integral:

$$\left|\int_a^b g(t)\sin pt\,dt\right| \leqslant \sum_{i=0}^{n-1}\omega_i \Delta t_i + \frac{2}{p}\sum_{i=0}^{n-1}|m_i|.$$

Taking an arbitrary number $\varepsilon > 0$, we choose the division (6) in advance in such a way that $\omega_i < \varepsilon/\{2(b-a)\}$, so that

$$\sum_{i=0}^{n-1} \omega_i \Delta t_i < \frac{\varepsilon}{2};$$

this can be done because of the continuity of the function g. Now, since the numbers m_i are already defined, for the same reason we can take

$$p > \frac{4}{\varepsilon} \sum |m_i|$$

and for these values of p we obtain

$$\left| \int_a^b g(t) \sin pt \, dt \right| < \varepsilon,$$

which proves our assertion.

We draw the reader's attention to the fact that here the limits between which the integral extends are established except for the passage to a limit under the integral sign.

If we recall the formulae (1) which express the Fourier coefficients, then as the first immediate corollary of this we obtain the assertion:

The Fourier coefficients a_m, b_m of a piece-wise continuous function tend to zero as $m \to \infty$.

401. The principle of localization. The second immediate corollary of the lemma we have proved is the so-called "principle of localization".

Taking an arbitrary positive number $\delta < \pi$, we divide up the integral in (4) into two parts: $\int_0^\pi = \int_0^\delta + \int_\delta^\pi$. If the second of these is rewritten in the form

$$\frac{1}{\pi} \int_\delta^\pi \frac{f(x_0+t) + f(x_0-t)}{2 \sin \frac{1}{2} t} \sin\left(n + \frac{1}{2}\right) t \, dt,$$

then it becomes clear that the factor with the sine

$$\frac{f(x_0+t) - f(x_0-t)}{2 \sin \frac{1}{2} t}$$

is a piece-wise continuous function of t in the interval $[\delta, \pi]$, because such a function of t, standing in the numerator, is continuous at the same time as the denominator $2\sin(t/2)$, which does not vanish in this interval. In this case, by the lemma, this integral tends to zero as $n \to \infty$, so that the existence of a limit for the partial sum of the Fourier series, $s_n(x_0)$, and the value of this limit are entirely determined by the behaviour of the one integral

$$\varrho_n(\delta) = \frac{1}{\pi} \int_0^{\delta} [f(x_0 + t) + f(x_0 - t)] \frac{\sin(n + \frac{1}{2})t}{2\sin\frac{1}{2}t}\, dt. \qquad (7)$$

But in this integral there are contained only the values of the function $f(x)$ corresponding to the change of the argument in the interval from $x_0 - \delta$ to $x_0 + \delta$. By these simple arguments we have proved the "principle of localization", stated in the following:

THEOREM. *The behaviour of the Fourier series of the function $f(x)$ at the point x_0† depends solely on the values taken by this function in the immediate proximity of the point under consideration*, i.e. in as small a neighbourhood of it as desired.

Thus if, for instance, we take two functions whose values in an arbitrarily small neighbourhood of x_0 coincide, then however much they differ outside this neighbourhood, the Fourier series corresponding to these functions behave the same at the point x_0: either they both converge, and to the same sum, or they both diverge. This result is shown to be even more remarkable if we emphasize that the actual Fourier coefficients of the functions under consideration, depending on all their values, may be quite different.

This theorem is usually connected with Riemann's name, because it is a corollary of his more general theorem, proved in 1835. It is necessary, however, to note that the idea of the "principle of localization" was already contained in a work by Ostrogradski on mathematical physics in 1828, and was also reflected in Lobatchevsky's investigations on trigonometrical series in 1834.

402. The representation of a function by Fourier series. We return to our interrupted investigation of the behaviour of the partial

† We understand by this the convergence or divergence of the series at the point x_0, and also—in the case of convergence—whether it has this or another sum.

sum $s_0(x_0)$ of the Fourier series, for which we obtained the integral representation (4).

Now we impose stricter requirements on the function $f(x)$, and in fact *we presuppose that it is piece-wise differentiable in the interval* $[-\pi, \pi]$†.

Then there holds the general

THEOREM. *If the function $f(x)$ with period 2π is piece-wise differentiable in the interval $[-\pi, \pi]$, then its Fourier series at each point $x = x_0$ converges and has the sum*

$$S_0 = \frac{f(x_0 + 0) + f(x_0 - 0)}{2}.$$

This sum, clearly, is equal to $f(x_0)$, if at the point $x = x_0$ the function is continuous.

Proof. We note that the equation (4) holds for every function $f(x)$ which satisfies the conditions stated. If, in particular, we take $f(x) \equiv 1$, then $s_n(x) \equiv 1$, and from (4) we obtain that

$$1 = \frac{2}{\pi} \int_0^{\pi} \frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{1}{2} t} \, dt. \tag{8}$$

Multiplying both sides of this equation by the constant number S_0 and subtracting the result from (4), we find

$$s_n(x_0) - S_0 = \frac{1}{\pi} \int_0^{\pi} [f(x_0 + t) + f(x_0 - t) - f(x_0 + 0) - f(x_0 - 0)] \frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{1}{2} t} \, dt;$$

for our purpose we need to prove that the integral on the right tends to zero as $n \to \infty$.

† A function $f(x)$ is called piece-wise differentiable in the interval $[a, b]$, if this interval can be divided up into a finite number of part intervals, inside which the function is differentiable, and at the end-points it not only has limit values, but also one-sided derivatives, under the condition that at these end points the values of the function are replaced by the limit values mentioned. A piece-wise differentiable function may be "put together" from several functions, differentiable (and hence also continuous) in closed part intervals, provided only that at the "junction points" (equally at the end-points a and b of the original interval) its values are shown to be singular.

We represent this in the form

$$\frac{1}{\pi}\int_0^{\pi} g(t)\sin(n+\tfrac{1}{2})t\,dt, \tag{9}$$

where it is supposed that

$$g(t)=\left[\frac{f(x_0+t)-f(x_0+0)}{t}-\frac{f(x_0-t)-f(x_0+0)}{-t}\right]\frac{\frac{1}{2}t}{\sin\frac{1}{2}t}; \tag{10}$$

if we were to succeed in establishing that this function is piece-wise continuous, then from the lemma in Sec. 400 it would follow that the integral (9) has the limit zero as $n\to\infty$. But in the interval $(0,\pi]$ the function $g(t)$ is in general continuous, with perhaps the exception of only a finite number of points, where it may have jumps—because f is such a function. There remains open only the question of the behaviour of the function $g(t)$ as $t\to+0$.

We will prove the existence of a finite limit

$$\lim_{t\to+0} g(t)=K;$$

setting then $g(0)=K$ (until now this value has remained indeterminate), we obtain continuity at the point $t=0$, and the application of the lemma is justified. But the second factor on the right side of equation (10) clearly has the limit one; we turn to the expression in the square brackets.

For simplicity, let the point x_0 first lie inside an interval where the function $f(x)$ is differentiable. Then $f(x_0+0)=f(x_0-0)=f(x_0)$, and each of the ratios

$$\frac{f(x_0+t)-f(x_0)}{t},\qquad \frac{f(x_0-t)-f(x_0)}{-t} \tag{11}$$

tends to the limit $f'(x_0)$, and [...] tends to zero. If, however, x_0 is a "junction point", then here it may be either a point of continuity or a point of discontinuity. In the first case we again encounter the ratios (11), but this time they will tend to different limits, to the right derivative and to the left derivative, respectively (Fig. 74*a*). We arrive at an analogous result in the case of a discontinuity, but here the values $f(x_0)$ are replaced by the values $f(x_0\pm0)$ of the functions from the "joining together" of which the given function was obtained,

and the limits of the ratios will be the one-sided derivatives of the given functions when $x = x_0$ (Fig. 74*b*).

Thus our conclusion is true in all cases.

403. The case of a non-periodic function. All the theory developed above started from the hypothesis that the given function is defined for all real values of x and also has period 2π. However, it is frequently necessary to discuss a non-periodic function $f(x)$, even though it is sometimes only given in the interval $[-\pi, \pi]$.

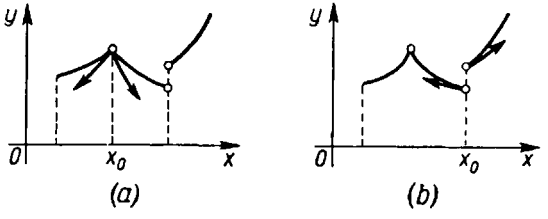

FIG. 74.

In order to be able to apply the theory we have developed to such a function, we introduce instead of it the auxiliary function $f^*(x)$, defined in the following way. In the interval $(-\pi, \pi]$ we identify f^* with f:

$$f^*(x) = f(x) \qquad (-\pi < x \leqslant \pi), \tag{12}$$

then we set

$$f^*(-\pi) = f^*(\pi) = f(\pi),$$

and we extend the function $f^*(x)$ to the remaining real values of x by the rule of periodicity.

The theory of expansions, which we have proved, can be applied to the function $f^*(x)$ constructed in this way, which has period 2π. However, if we are discussing a point x_0, lying strictly between $-\pi$ and π, then, in view of (12), we only need to discuss the given function $f(x)$. For the same reason the coefficients of the expansion can be evaluated by formula (1), without turning to the function $f^*(x)$. To put it more briefly, *all that has been proved above carries over immediately to the given function* $f(x)$, without using the auxiliary function $f^*(x)$.

The end-points of the interval, however, $x = \pm\pi$, require particular attention. In applying the theorem of Sec. 402 to the function $f^*(x)$, say, at the point $x = \pi$, we need to discuss both the values of the auxiliary function $f^*(x)$ on the left of $x = \pi$, where they coincide with the corresponding values of the given function $f(x)$, and the values of $f^*(x)$ on the right of $x = \pi$, *where they coincide with the values of* $f(x)$ *on the right of* $x = -\pi$. Therefore for $x = \pm\pi$ as the value of S_0 it is necessary to take

$$S_0 = \frac{f^*(\pi+0)+f^*(\pi-0)}{2} = \frac{f^*(-\pi+0)+f^*(-\pi-0)}{2}$$

$$= \frac{f(-\pi+0)+f(\pi-0)}{2}.$$

Thus, even if the given function $f(x)$ is continuous at $x = \pm\pi$, but does not have period 2π, so that $f(\pi) \neq f(-\pi)$, then—if the requirement of piece-wise differentiability is satisfied—the sum of the Fourier series will be

$$\frac{f(-\pi)+f(\pi)}{2},$$

which is equal neither to $f(-\pi)$ nor to $f(\pi)$. For such a function the expansion can only hold in the open interval $(-\pi, \pi)$.

The following remark deserves the reader's serious attention. If the trigonometrical series (2) converges in the interval $(-\pi, \pi]$ to the function $f(x)$, then by virtue of the fact that its terms have period 2π, it converges everywhere, and its sum $S(x)$ is also a periodic function of x with period 2π. *But this sum outside the given interval does not, in general, coincide with the function* $f(x)$ (if the latter is given on the whole real axis). Below, this remark will be illustrated by examples.

We note, finally, that instead of the interval $[-\pi, \pi]$ we could take any interval $[\alpha, \alpha + 2\pi]$ of length 2π.

404. The case of an arbitrary interval. We presuppose that the function $f(x)$ is given in an interval $[-l, l]$ of arbitrary length $2l$ and is piece-wise differentiable in it. If we make use of the substitution

$$x = \frac{ly}{\pi} \qquad (-\pi \leqslant y \leqslant \pi),$$

then we obtain the function $f(ly/\pi)$ of y in the interval $[-\pi, \pi]$, also piece-wise differentiable, to which all the considerations of the preceding section are applicable. As we have seen, with the exclusion of the points of discontinuity and the end-points $-\pi, \pi$ of the interval, we can expand it in a Fourier series:

$$f\left(\frac{ly}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos ny + b_n \sin ny),$$

whose coefficients are defined by the Euler–Fourier formulae:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{ly}{\pi}\right) \cos ny \, dy, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{ly}{\pi}\right) \sin ny \, dy.$$

$$(n = 0, 1, 2, \ldots) \qquad (n = 1, 2, 3, \ldots)$$

Now we return to the original variable x, letting

$$y = \frac{\pi x}{l}.$$

Then we obtain an expansion of the given function $f(x)$ in a trigonometrical series of some known type:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}\right). \tag{13}$$

Here the cosines and sines are of angles which are multiples, not of x, but of $\pi x/l$. We can also transform the formulae for determining the coefficients of this expansion by the same substitution to the form

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, dx, \quad b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx. \tag{14}$$

$$(n = 0, 1, 2, \ldots) \qquad (n = 1, 2, 3, \ldots)$$

With respect to the end-points of the interval $x = \pm l$, the remarks made in the preceding section about the points $x = \pm\pi$ still hold. Finally, the interval $(-l, l]$ can be replaced by any other interval

of length $2l$, in particular, by the interval $(0, 2l]$. In this case the formulae (14) must be replaced by the formulae

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx, \qquad b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx. \quad (14a)$$

$$(n = 0, 1, 2, \ldots) \qquad (n = 1, 2, 3, \ldots)$$

405. An expansion in cosines only, or in sines only. We begin with the following remark: if there is given in the interval $[-\pi, \pi]$ the (continuous or, at least, piece-wise continuous) function $f(x)$, which is odd, then for it

$$\int_{-\pi}^{\pi} f(x)\,dx = 0.$$

This is easy to ascertain, by representing the integral $\int_{-\pi}^{\pi}$ in the form of the sum of the integrals: $\int_0^{\pi} + \int_{-\pi}^{0}$ and replacing x by $-x$ in the second of them. In the same way we establish that in the case of an even function $f(x)$:

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\int_0^{\pi} f(x)\,dx.$$

Now let $f(x)$ be a piece-wise differentiable even function in the interval $[-\pi, \pi]$. Then the product $f(x)\sin nx$ is an odd function, and by what we have said

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0.$$

Thus, *the Fourier series of an even function contains only cosine terms*:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx. \qquad (15)$$

Because $f(x)\cos nx$ in this case will be an even function, then, applying here the second of the remarks made above, we can rewrite the coefficients a_n of the expansion in the form

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \qquad (n = 0, 1, 2, \ldots). \qquad (16)$$

If, however, the function $f(x)$ is odd, then the function $f(x)\cos nx$ will also be odd, so that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 \qquad (n = 0, 1, 2, \ldots).$$

We arrive at the conclusion that *the Fourier series of an odd function contains only sine terms*:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx. \tag{17}$$

Here, in view of the fact that the product $f(x)\sin nx$ is even, we can write:

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx \qquad (n = 1, 2, 3, \ldots). \tag{18}$$

We note in passing that every function $f(x)$ given in the interval $[-\pi, \pi]$ can be represented in the form of a sum of even and odd component functions:

$$f(x) = f_1(x) + f_2(x),$$

where

$$f_1(x) = \frac{f(x) + f(-x)}{2}, \qquad f_2(x) = \frac{f(x) - f(-x)}{2}.$$

It is obvious that the Fourier series of the function $f(x)$ is composed precisely of the expansion in cosine terms of the function $f_1(x)$ and the expansion in sine terms of the function $f_2(x)$.

We presuppose, further, that the function $f(x)$ is given only in the interval $[0, \pi]$. Wishing to expand it in this interval in a Fourier series (2), we supplement the definition of our function for values of x in the interval $[-\pi, 0)$ arbitrarily, but keeping the piece-wise differentiability, and then we apply what was said in Sec. 403. The arbitrariness emphasized above in the definition of the function makes it possible to obtain in this way different trigonometrical series.

We can use the arbitrariness in the definition of the function in the interval $[-\pi, 0)$ in such a way that we obtain an expansion

for $f(x)$ only in cosine terms or only in sine terms. In fact, for $0 < x \leqslant \pi$ we set

$$f(-x) = f(x), \tag{19}$$

so that we obtain an even function in the interval $[-\pi, \pi]$ (Fig. 75*a*) in which it has period 2π. Its expansion, as we have seen, will contain only cosine terms. The coefficients of the expansion can be evaluated by formulae (16), where only values of the function $f(x)$ originally given occur.

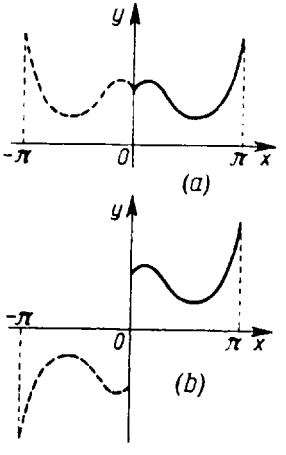

FIG. 75.

Analogously, if we supplement the definition of the function by the condition (for $0 < x \leqslant \pi$)

$$f(-x) = -f(x) \tag{20}$$

so that it is odd (Fig. 75*b*), then only terms involving sines will occur in its expansion. Its coefficients are determined by the formulae (18).

Thus, *when known conditions are satisfied, a function given in the interval* $[0, \pi]$ *can be expanded both in a series of cosine terms, and in a series of sine terms.*

Special investigation, however, is required at the points $x = 0$ and $x = \pi$. Here the two expansions behave differently. For simplicity we presuppose that the given function $f(x)$ is continuous when

$x = 0$ and $x = \pi$, and we consider at first the expansion in cosine terms. The condition (19) is continuous when $x = 0$, so that the series (15), when $x = 0$, will converge to $f(0)$. Because, further,

$$f(-\pi + 0) = f(\pi - 0) = f(\pi),$$

an analogous situation holds when $x = \pi$.

The situation is different with the expansion in sine terms. Without entering into considerations as to the violation of continuity by the condition (20) and so on, we simply note that at the points $x = 0$ and $x = \pi$ the sum of the series (17) will clearly be equal to zero. Therefore it can give us the values of $f(0)$ and $f(\pi)$, clearly, only in the case when these values are also equal to zero.

If the function $f(x)$ is given in the interval $[0, l]$ $(l > 0)$, then, using the same change of variables as we used in Sec. 404, we reduce the problem of expanding it in a series of cosine terms

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

or in a series of sine terms

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

to the case which we have just considered. Here the coefficients of the expansion can be evaluated, respectively, by means of the formulae

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} \, dx \qquad (n = 0, 1, 2, \ldots) \tag{21}$$

or

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx \qquad (n = 1, 2, 3, \ldots). \tag{22}$$

406. Examples. The functions, which are introduced below in the form of examples, as a rule, belong to the class of the differentiable or piece-wise differentiable functions. Therefore the actual possibility of expanding them in Fourier series is in no doubt, and we will not linger over this question.

(1) To expand the function

$$f(x) = \frac{\pi - x}{2}$$

in the interval $(0, 2\pi)$.

By the formulae (1a), Sec. 399:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - x}{2}\,dx = \frac{1}{2\pi}\left(\pi x - \frac{1}{2}x^2\right)\Big|_0^{2\pi} = 0,$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - x}{2}\cos nx\,dx = \frac{1}{2\pi}(\pi - x)\,\frac{\sin nx}{n}\Big|_0^{2\pi} + \frac{1}{2n\pi}\int_0^{2\pi} \sin nx\,dx = 0,$$

$$(n = 1, 2, 3, \ldots)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - x}{2}\sin nx\,dx = -\frac{1}{2\pi}(\pi - x)\,\frac{\cos nx}{n}\Big|_0^{2\pi} - \frac{1}{2n\pi}\int_0^{2\pi} \cos nx\,dx = \frac{1}{n}.$$

Thus we arrive at the remarkably simple expansion, containing only sine terms:

$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{\sin nx}{n} \qquad (0 < x < 2\pi).$$

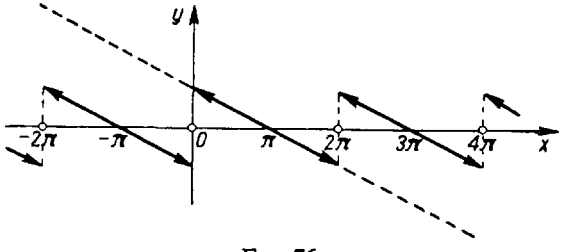

FIG. 76.

When $x = 0$ (or 2π) the sum of the series is equal to zero and the equation is violated. The equation will not hold outside the given interval. The graph of the sum of the series, $S(x)$ (Fig. 76), consists of an infinite set of parallel segments and a series of separate points on the x-axis.

(2) From the expansion in *(1)* without further calculation we can obtain other interesting expansions. Replacing x by $2x$ in it and dividing both sides of the equation by 2, we find:

$$\frac{\pi}{4}-\frac{x}{2}=\sum_{k=1}^{\infty}\frac{\sin 2kx}{2k} \qquad (0<x<\pi);$$

subtracting one expansion from the other, we obtain

$$\frac{\pi}{4}=\sum_{k=1}^{\infty}\frac{\sin(2k-1)x}{2k-1} \qquad (0<x<\pi).$$

If by $S(x)$ we denote the sum of the last series, then $S(0)=S(\pi)=0$. Changing the sign of x, for the interval $(-\pi, 0)$ we find, since the sine function is an odd function, that $S(x)=-\pi/4$; for the other values of x the sum $S(x)$ is obtained by the rule of periodicity, so that, in particular, for the interval $(2\pi, 3\pi)$ again $S(x)=\pi/4$, and so on. The graph of the function $S(x)$ is illustrated in Fig. 77; Fig. 78 illustrates the gradual approximation to this discontinuous function made by the partial sums of the series.

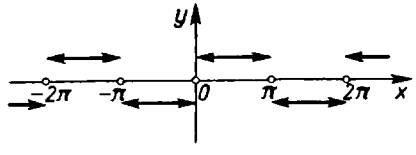

FIG. 77.

Remark. If we set $x=\pi/2$ in the expansion we have been considering, then we obtain Leibniz's series, which is already well known to us [Sec. 255, (20)]:

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots.$$

Combining the expansion obtained here with the expansion in *(1)*, it is easy to arrive at a series for the function

$$x=2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sin nx}{n} \qquad (-\pi<x<\pi).$$

We obtain it immediately only for $0<x<\pi$, but the equation clearly holds for $x=0$ and, moreover, both sides of it, obviously, represent odd functions, so that the final expansion is true for the whole interval $(-\pi, \pi)$.

The graph of the sum of the series as x varies from $-\infty$ to $+\infty$ is shown in Fig. 79. In Fig. 80 we show the graph of the partial sum

$$y=s_5(x)=2\left(\sin x-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\frac{\sin 4x}{4}+\frac{\sin 5x}{5}\right).$$

(3) To expand:

(a) the even function $f_1(x) = \cos ax$ by cosine terms in $[-\pi, \pi]$,

(b) the odd function $f_2(x) = \sin ax$ by sine terms in $(-\pi, \pi)$ (the number a is here presupposed not an integer).

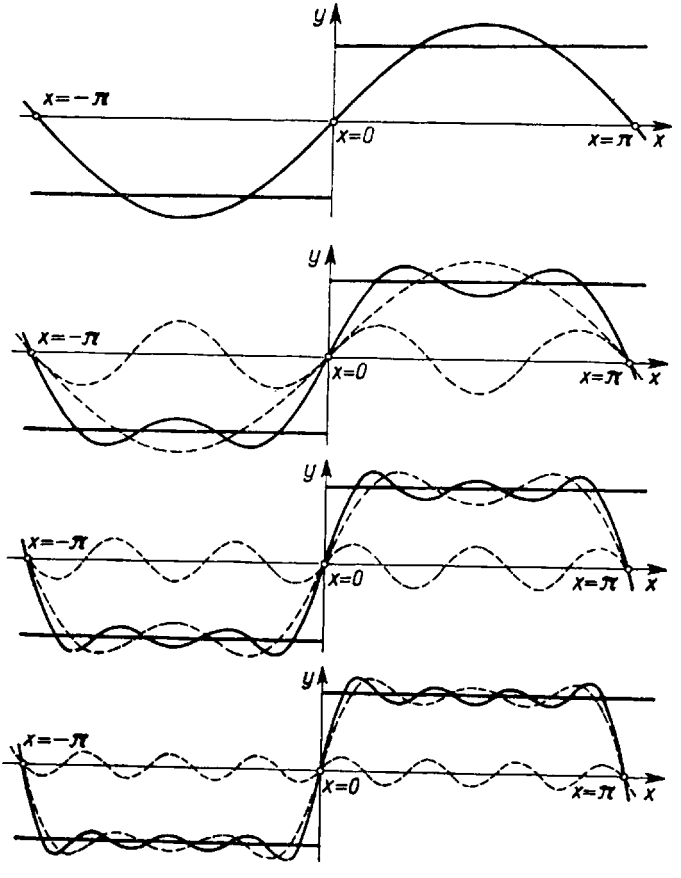

FIG. 78.

(a) We have

$$\frac{1}{2} a_0 = \frac{1}{\pi} \int_0^{\pi} \cos ax\, dx = \frac{\sin a\pi}{a\pi}$$

$$(n>0) \quad a_n = \frac{2}{\pi}\int_0^{\pi} \cos ax \cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} [\cos(a+n)x + \cos(a-n)x]\, dx = (-1)^n \frac{2a}{a^2-n^2}\frac{\sin a\pi}{\pi},$$

so that

$$\frac{\pi}{2}\frac{\cos ax}{\sin a\pi} = \frac{1}{2a} + \sum_{n=1}^{\infty} (-1)^n \frac{a\cos nx}{a^2-n^2} \qquad (-\pi < x < \pi).$$

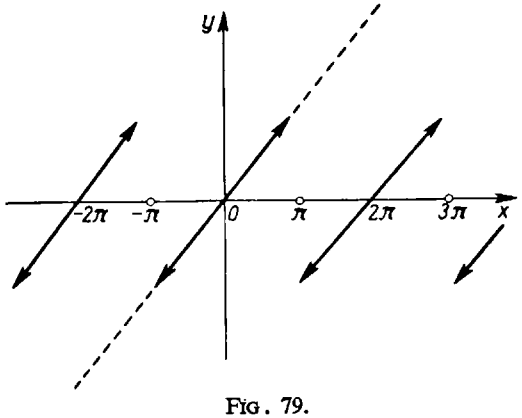

FIG. 79.

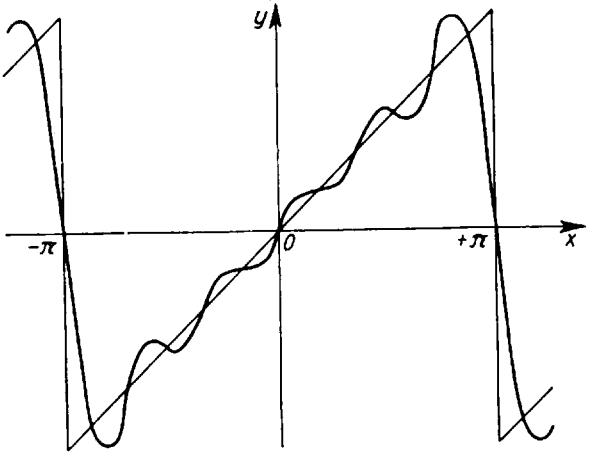

FIG. 80.

(b) *Answer.*

$$\frac{\pi}{2}\frac{\sin ax}{\sin a\pi} = \sum_{n=1}^{\infty}(-1)^n \frac{n\sin nx}{a^2-n^2} \qquad (-\pi < x < \pi).$$

Remark. We note in passing that when $x = 0$ we obtain from (a):

$$\frac{1}{\sin a\pi} = \frac{1}{a\pi} + 2\sum_{n=1}^{\infty}(-1)^n \frac{a\pi}{(a\pi)^2-(n\pi)^2}$$

or, if we set $a\pi = z$:

$$\frac{1}{\sin z} = \frac{1}{z} + \sum_{n=1}^{\infty}(-1)^n \frac{2z}{z^2-(n\pi)^2}$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty}(-1)^n \left[\frac{1}{z-n\pi} + \frac{1}{z+n\pi}\right]$$

(here z is any number which is not a multiple of π). We have arrived at the "expansion in simple fractions" of the function $1/\sin z$ (which we have already used in Sec. 293, *(3)* and 308, *(1)*). By setting $x = \pi$ in (a) we can obtain the "expansion in simple fractions" of the function $\cot z$:

$$\cot z = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2-(n\pi)^2}$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty}\left[\frac{1}{z-n\pi} + \frac{1}{z+n\pi}\right].$$

(4) To expand the function

$$f(x) = \frac{x^2}{4} - \frac{\pi x}{2}$$

by cosine terms in the interval $[0, \pi]$.

It is easy to calculate, by the formulae (16):

$$\frac{a_0}{2} = \frac{1}{\pi}\int_0^{\pi}\left(\frac{x^2}{4} - \frac{\pi x}{2}\right)dx = -\frac{\pi^2}{6}$$

and further (integrating twice by parts)

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(\frac{x^2}{4}-\frac{\pi x}{2}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left(\frac{x^2}{4}-\frac{\pi x}{2}\right)\frac{\sin nx}{n}\Bigg|_0^{\pi} - \frac{1}{n\pi}\int_0^{\pi}(x-\pi)\sin nx\,dx$$

$$= \frac{1}{n^2\pi}(x-\pi)\cos nx\Bigg|_0^{\pi} - \frac{1}{n^2\pi}\int_0^{\pi}\cos nx\,dx = \frac{1}{n^2}\ (n = 1, 2, 3, \ldots).$$

The expansion

$$\frac{x^2}{4}-\frac{\pi x}{2} = -\frac{\pi^2}{6}+\sum_{n=1}^{\infty}\frac{\cos nx}{n^2},$$

derived for the interval $[0, \pi]$, in fact holds even in the interval $[0, 2\pi]$, in so far as both sides of the equation do not change sign when x is replaced by $2\pi - x$.

Remark. If we set $x = 0$ here, then we obtain the well-known Euler series:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

407. The expansion of a continuous function in a series of trigonometrical polynomials. In the proof of the theorem on expansion in Fourier series [Sec. 402] for simplicity we limited ourselves to piece-wise differentiable functions. In fact, the class of functions which admit such an expansion is considerably wider, but nevertheless not all continuous functions with period 2π are contained in it. [See Sec. 424.] In a certain sense a situation is created here similar to that which was discussed in Sec. 278 regarding the expansion of a function in a power series. Together with the theorem of Weierstrass (given there) on the expansion of an arbitrary continuous function in a uniformly convergent series of algebraic polynomials, the same author proved a second theorem—on the expansion of an arbitrary continuous function with period 2π in a uniformly convergent series of trigonometrical polynomials, i.e. of a sum of the form

$$A_0 + \sum_{k=1}^{m}(A_k\cos kx + B_k\sin kx). \tag{23}$$

Thus, as above, the extension of the domain of applicability of the expansion is achieved the complicated character of the actual terms of the expansion: instead of monomial expressions of simple form there are polynomial sums of them.

We formulate the second theorem of Weierstrass, like the first, in the language of sequences:

WEIERSTRASS'S THEOREM. *If the function $f(x)$ is continuous in the interval $[-\pi, \pi]$ and has period 2π:*

$$f(-\pi) = f(\pi),$$

then there exists a sequence of trigonometrical polynomials which in this interval converges uniformly to $f(x)$†.

We begin the proof of the theorem with the simple remark that the product of two trigonometrical polynomials of type (23) can also be represented in the form of a trigonometrical polynomial. This follows immediately from the fact that all the possible products

$$\cos kx \cdot \cos lx,\ \cos kx \cdot \sin lx,\ \sin kx \cdot \sin lx$$

by well-known formulae of trigonometry can be represented in the form of such polynomials, for instance,

$$\cos kx \cdot \cos lx = \frac{1}{2}\cos(k+l)x + \frac{1}{2}\cos(k-l)x.$$

The theorem will be proved if we establish that for any number $\varepsilon > 0$ we can find a trigonometrical polynomial $T(x)$ such that for all x in the interval $[-\pi, \pi]$ it will be true that

$$|f(x) - T(x)| < \varepsilon. \tag{24}$$

This is easy to establish for the case of an *even function* $f(x)$. By using Weierstrass's first theorem [Sec. 278], we first of all construct an algebraic polynomial $P(t)$ such that for all values of t in the interval $[-1, 1]$ it will be true that

$$|f(\arccos t) - P(t)| < \varepsilon‡.$$

Here setting $t = \cos x$, where $0 \leqslant x \leqslant \pi$, we obtain for the whole of this interval

$$|f(x) - P(\cos x)| < \varepsilon.$$

Since the functions $f(x)$ and $\cos x$ are even, this inequality remains true when x is replaced by $-x$, i.e. it will be satisfied for all values of x in the interval $[-\pi, \pi]$, and by virtue of the periodicity it will also be satisfied in the whole interval $(-\infty, +\infty)$. Because, by the remark made at the beginning, the expression $P(\cos x)$ can be written in the form of a trigonometrical polynomial, for the case under consideration our assertion is true.

Now we turn to the general case—*an arbitrary continuous function* $f(x)$ *with period* 2π. For even functions

$$f_1(x) = \frac{f(x) + f(-x)}{2}, \quad f_2(x) = \frac{f(x) - f(-x)}{2}\sin x;$$

by what has been proved, there exist trigonometrical polynomials $T_1(x)$ and $T_2(x)$ such that

$$|f_1(x) - T_1(x)| < \frac{\varepsilon}{4}, \quad |f_2(x) - T_2(x)| < \frac{\varepsilon}{4} \quad (-\infty < x < +\infty).$$

---

† We will assume that the function $f(x)$ is extended periodically over the whole interval $(-\infty, +\infty)$.

‡ We recall that the values of arccos t fill the interval $[0, \pi]$.

Hence it follows that

$$|f_1 \sin^2 x - T_1 \sin^2 x| < \frac{\varepsilon}{4}, \quad |f_2 \sin x - T_2 \sin x| < \frac{\varepsilon}{4},$$

and then, if we add these inequalities:

$$|f(x) \sin^2 x - T_3(x)| < \frac{\varepsilon}{2}, \tag{25}$$

where $T_3(x)$ is the trigonometrical polynomial

$$T_3 = T_1 \sin^2 x + T_2 \sin x$$

[see the Remark].

In a similar way, for the function $f(x + (\pi/2))$ we find a trigonometrical polynomial $T_4(x)$ such that

$$\left| f\left(x + \frac{\pi}{2}\right) \sin^2 x - T_4(x) \right| < \frac{\varepsilon}{2}.$$

Replacing x by $x - (\pi/2)$ here and denoting by $T_5(x)$ the trigonometrical polynomial which is thus obtained from $T_4(x)$, we find that

$$|f(x) \cos^2 x - T_5(x)| < \frac{\varepsilon}{2}. \tag{26}$$

Finally, from (25) and (26) by addition we obtain (24), by letting $T = T_3 + T_5$. The theorem is proved.

We can also formulate this theorem in the language of series [cf. Sec. 278]:

A continuous function $f(x)$ with period 2π can be expanded in a series uniformly convergent to it, consisting of trigonometrical polynomials.

§ 3. The Fourier integral

408. The Fourier integral as a limiting case of a Fourier series. We wish to reproduce here in essence the considerations, remarkable for their clarity, although lacking in rigorousness, which were used by Fourier in obtaining his integral formula†.

If the function $f(x)$ is given in a finite interval $[-l, l]$, then under definite conditions, which do not interest us here, it can be represented in this interval by a trigonometrical series:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos \frac{m\pi x}{l} + b_m \sin \frac{m\pi x}{l},$$

where

$$a_m = \frac{1}{l} \int_{-l}^{l} f(u) \cos \frac{m\pi u}{l}\, du, \quad (m = 0, 1, 2, \ldots) \qquad b_m = \frac{1}{l} \int_{-l}^{l} f(u) \sin \frac{m\pi u}{l}\, du \quad (m = 1, 2, \ldots)$$

† This formula was obtained by Cauchy independently of Fourier.

[see Sec. 404]. Substituting the expressions for the coefficients a_m and b_m, we can rewrite the series in the form

$$f(x) = \frac{1}{2l}\int_{-l}^{l} f(u)\,du + \sum_{m=1}^{\infty}\frac{1}{l}\int_{-l}^{l} f(u)\cos\frac{m\pi}{l}(u-x)\,du. \tag{1}$$

Now let the function $f(x)$ be defined in the whole of the infinite interval $(-\infty, +\infty)$. In this case, whatever x may be, the corresponding value of $f(x)$ is expressed by the expansion (1) for any $l > |x|$. Taking limits here as $l \to +\infty$, we try to establish the "limit form" of this expansion.

It is natural to assume that the first term on the right side of equation (1) tends to zero†. Turning to the infinite series, we can consider the factors $m\pi/l$ under the sign of the cosine as discrete values

$$z_1 = \frac{\pi}{l}, \quad z_2 = \frac{2\pi}{l}, \quad \ldots, \quad z_m = \frac{m\pi}{l}, \quad \ldots$$

of a variable z, which varies continuously from $z_0 = 0$ to $+\infty$; here the increment

$$\Delta z_m = z_{m+1} - z_m = \frac{\pi}{l}$$

clearly tends to zero as $l \to +\infty$. In this notation our series can be rewritten thus:

$$\frac{1}{\pi}\sum_{m=1}^{\infty}\Delta z_{m-1}\int_{-l}^{l} f(u)\cos z_m(u-x)\,du.$$

It recalls the integral sum for the function

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} f(u)\cos z(u-x)\,du$$

of z in the interval $[0, +\infty]$. Taking limits as $l \to +\infty$, instead of a series we obtain an integral; in this way we arrive at Fourier's integral formula:

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty} dz \int_{-\infty}^{+\infty} f(u)\cos z(u-x)\,du.$$

We represent this formula, which contains an expression for the cosine of a difference, in the form

$$f(x) = \int_{0}^{\infty}[a(z)\cos zx + b(z)\sin zx]\,dz,$$

---

† This becomes obvious, for instance, if we presuppose that the integral $\int_{-\infty}^{+\infty} f(u)\,du$ converges.

where

$$a(z)=\frac{1}{\pi}\int_{-\infty}^{+\infty} f(u)\cos zu\,du, \qquad b(z)=\frac{1}{\pi}\int_{-\infty}^{+\infty} f(u)\sin zu\,du.$$

Here it is easy to detect an analogy with the trigonometrical expansion: only the parameter m, which occurs in the series of natural values, is replaced here by a continuously varying parameter z, and the infinite series is replaced by an integral. The coefficients $a(z)$ and $b(z)$ in their structure also represent the Fourier coefficients.

Of course, all these considerations are only in the nature of a guide; the actual conditions for the validity of Fourier's formula still need to be clarified. But in carrying out the rigorous argument we will follow the fundamental steps connected with Fourier series.

409. Preliminary remarks. *With respect to the function $f(x)$ we now presuppose that (1) it is piece-wise continuous in every finite interval and (2) it is absolutely integrable in the infinite interval* $[-\infty, +\infty]$. With these hypotheses we consider the integral

$$J(A)=\frac{1}{\pi}\int_0^A dz\int_{-\infty}^{+\infty} f(u)\cos z(u-x_0)\,du,$$

where A is an arbitrary finite positive number, and x_0 is any fixed value of x. This integral represents an analogy with the partial sum of a Fourier series: the Fourier integral of it

$$\frac{1}{\pi}\int_0^{+\infty} dz\int_{-\infty}^{+\infty} f(u)\cos z(u-x_0)\,du \tag{2}$$

is obtained as the limit when $A\to+\infty$.

For any finite $B>0$, by Theorem 4 of Sec. 298, we will have

$$\int_0^A dz\int_{-B}^{B} f(u)\cos z(u-x_0)\,du=\int_{-B}^{B} f(u)\,du\int_0^A \cos z(u-x_0)\,dz$$

$$=\int_{-B}^{B} f(u)\frac{\sin A(u-x_0)}{u-x_0}\,du. \tag{3}$$

This follows immediately from the theorem mentioned, if the function $f(u)$ is continuous in the interval $[-B, B]$; in the contrary case the theorem must be applied separately to each of the intervals in which the function is continuous.

But the integral

$$\int_{-\infty}^{+\infty} f(u)\cos z(u-x_0)\,du \tag{4}$$

is exceeded by the integral

$$\int_{-\infty}^{+\infty} |f(u)|\,du$$

which converges by hypothesis and, hence, it converges uniformly with respect to z (both when $u = +\infty$ and when $u = -\infty$) for any interval of its values. Thus the integral

$$\int_{-B}^{B} f(u)\cos z(u - x_0)\,du$$

tends to its limit (4) uniformly as $B \to +\infty$. Therefore, in taking limits in equation (3) as $B \to +\infty$, in the integral on the left limits can be taken under the integral sign [Theorem 1, Sec. 296]†. Hence for $J(A)$ we obtain the expression in the form of an integral

$$J(A) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(u)\frac{\sin A(u - x_0)}{u - x_0}\,du,$$

which recalls Dirichlet's integral [Sec. 399] and, in fact, plays exactly the same part. It is easy to make an elementary transformation of this to the form

$$J(A) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x_0 + t)\frac{\sin At}{t}\,dt$$

$$= \frac{1}{\pi}\int_{0}^{\infty} [f(x_0 + t) + f(x_0 - t)]\frac{\sin At}{t}\,dt. \tag{5}$$

For our further investigations we need the following obvious supplement to the fundamental lemma in Sec. 400:

If the function $g(t)$ is piece-wise continuous in every finite part of the interval $[a, +\infty]$ and is absolutely integrable in this infinite interval, then

$$\lim_{p \to +\infty} \int_{a}^{+\infty} g(t)\sin pt\,dt = 0$$

(and so equally

$$\lim_{p \to +\infty} \int_{a}^{+\infty} g(t)\cos pt\,dt = 0).$$

In fact, choosing an arbitrary number $\varepsilon > 0$, we first choose A large enough that

$$\int_{A}^{+\infty} |g(t)|\,dt < \frac{\varepsilon}{2},$$

† Cf. Theorem 4, Sec. 305, where we presupposed the function under the integral to be continuous. Here we make no such supposition.

and this means that certainly

$$\left| \int_A^{+\infty} g(t)\sin pt\, dt \right| \leqslant \int_A^{+\infty} |g(t)|\, dt < \frac{\varepsilon}{2},$$

and, moreover, for all p. But then, we can immediately apply to the integral

$$\int_0^A g(t)\sin pt\, dt$$

the lemma in Sec. 400, so that for sufficiently large p it will be true that

$$\left| \int_a^A g(t)\sin pt\, dt \right| < \frac{\varepsilon}{2};$$

for the same values of p, clearly,

$$\left| \int_a^{\infty} g(t)\sin pt\, dt \right| < \varepsilon,$$

which is what we wished to prove.

410. The representation of a function by a Fourier integral. A complete analogy with the theorem in Sec. 402 holds here.

THEOREM. *Let the function $f(x)$ be piece-wise differentiable in every finite interval and, at the same, absolutely integrable in the interval $[-\infty, +\infty]$. Then at every point $x = x_0$ its Fourier integral converges and has the value*

$$S_0 = \frac{f(x_0+0)+f(x_0-0)}{2}$$

(which, clearly, tends to $f(x_0)$ if the function is continuous at the point $x = x_0$).

Proof. Multiplying both sides of the equation

$$1 = \frac{2}{\pi}\int_0^{\infty} \frac{\sin At}{t}\, dt \qquad (A>0)$$

[Sec. 293, Remark; Sec. 308, *(2)*] by the constant number S_0, we subtract the result obtained from equation (5); we obtain

$$J(A) - S_0 = \frac{1}{\pi}\int_0^{\infty} [f(x_0+t)+f(x_0-t)-f(x_0+0)-f(x_0-0)]\, \frac{\sin At}{t}\, dt. \quad (6)$$

Taking a constant number $h>0$, we rewrite this integral in the form

$$\frac{1}{\pi}\int_0^h \left[\frac{f(x_0+t)-f(x_0+0)}{t} - \frac{f(x_0-t)-f(x_0-0)}{-t}\right] \sin At\, dt$$

$$+\frac{1}{\pi}\int_h^{\infty} \frac{f(x_0+t)+f(x_0-t)}{t} \sin At\, dt - \frac{2S_0}{\pi}\int_h^{\infty} \frac{\sin At}{t}\, dt.$$

The fact that the first integral tends to zero as $A \to \infty$ is established in the same way as in the proof of the theorem in Sec. 402. In the second integral the fractional expression, together with f, represents a piece-wise continuous function in any finite part of the interval $[h, \infty]$; at the same time this function is absolutely integrable in the given interval. Hence, by the remark at the end of the preceding section, the second integral also has the limit zero as $A \to \infty$. Finally, letting $At = z$ in the last integral, we reduce it to the form

$$-\frac{2S_0}{\pi} \int_{Ah}^{\infty} \frac{\sin z}{z}\, dz,$$

so that as $A \to \infty$ it also tends to zero. Hence we deduce that the expression (6) has the limit zero as $A \to \infty$, i.e. the integral (2) exists and is equal to S_0.

411. Different forms of Fourier's formula. Presupposing that the conditions mentioned above for the applicability of Fourier's formula are satisfied, we will assume for simplicity that at the point considered the function $f(x)$ is continuous or, if it is discontinuous, that it satisfies the condition

$$f(x) = \frac{f(x+0)+f(x-0)}{2}.$$

Then in all cases we have

$$f(x) = \frac{1}{\pi} \int_0^{+\infty} dz \int_{-\infty}^{+\infty} f(u)\cos z(u-x)\, du. \tag{7}$$

In view of the fact that the inner integral is clearly an even function of z, this formula can be rewritten thus:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dz \int_{-\infty}^{+\infty} f(u)\cos z(u-x)\, du. \tag{8}$$

It is easy to show further that under the general hypotheses made in Sec. 410 with respect to the function $f(x)$ the integral

$$\int_{-\infty}^{+\infty} f(u)\sin z\,(u-x)\, du \tag{9}$$

also exists. This integral, also, is a continuous function of z and clearly is odd.

Generally speaking, for this function it is impossible to guarantee the existence of the improper integral from $-\infty$ to $+\infty$.

In the case when for any function $\varphi(z)$ the integral from $-\infty$ to $+\infty$ does not exist, i.e. the limit

$$\lim_{\substack{M \to \infty \\ N \to \infty}} \int_{-M}^{N} \varphi(z)\, dz$$

as M and N tend independently to infinity does not exist, yet there may exist the limit corresponding to the special hypothesis $M = N$. Following Cauchy, we call such a limit *the principal value of the integral* and we denote it by the letters V.p. (Valeur principale):

$$\text{V.p.} \int_{-\infty}^{+\infty} \varphi(z)dz = \lim_{M\to\infty} \int_{-M}^{M} \varphi(z)dz.$$

If the integral exists in accordance with the usual definition of an improper integral, then, obviously, it coincides with the principal value.

In view of the fact that the function (9) of z is odd, we will have

$$\int_{-M}^{M} dz \int_{-\infty}^{+\infty} f(u)\sin z(u-x)du = 0,$$

and as $M\to\infty$ we again obtain the limit zero. Thus, in every case

$$\text{V.p.} \int_{-\infty}^{+\infty} dz \int_{-\infty}^{+\infty} f(u)\sin z(u-x)du = 0.$$

Multiplying this equation by $i/(2\pi)$ and adding it to (8), we arrive at the relation

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dz \int_{-\infty}^{+\infty} f(u)e^{iz(u-x)}du, \tag{10}$$

where the outer integral is understood in the sense of a principal value. In this form the formula was first presented by Cauchy.

Returning to formula (7), we rewrite it in the form

$$f(x) = \frac{1}{\pi}\int_0^{\infty} \cos zx\,dx \int_{-\infty}^{+\infty} f(u)\cos zu\,du + \frac{1}{\pi}\int_0^{\infty} \sin zx\,dz \int_{-\infty}^{+\infty} f(u)\sin zu\,du.$$

If $f(u)$ is an even function, then

$$\int_{-\infty}^{+\infty} f(u)\cos zu\,du = 2\int_0^{\infty} f(u)\cos zu\,du, \qquad \int_{-\infty}^{+\infty} f(u)\sin zu\,du = 0,$$

and we obtain a simplified formula, containing only cosine terms:

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \cos zx\,dx \int_0^{\infty} f(u)\cos zu\,du. \tag{11}$$

Analogously, in the case of an odd function $f(x)$ we arrive at a formula containing only sine terms:

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \sin zx\,dx \int_0^{\infty} f(u)\sin zu\,du. \tag{12}$$

Now let the function $f(x)$ be given only in the interval $[0, +\infty]$ and let it satisfy in the interval conditions analogous to those which were earlier set down with respect to the whole interval $(-\infty, +\infty)$. Then, by extending the function $f(x)$ to the interval $(-\infty, 0)$ with the help of the equations $(x > 0)$:

$$f(-x) = f(x) \quad \text{or} \quad f(-x) = -f(x),$$

we obtain in the first case an even, and in the second an odd, function in the interval $(-\infty, +\infty)$. For positive values of x we can thus use both formula (11) and formula (12).

If we presuppose the function $f(x)$ is continuous at the point $x = 0$, then the formula (11) is also applicable at this point, because the function continued as an even function is continuous there. However, formula (12) in general is not applicable at the point $x = 0$: it gives the value of $f(0)$ only in the case when this value is zero.

These facts are completely analogous to those mentioned in Sec. 405 on Fourier series.

412. Fourier transforms. Under our hypotheses Fourier's formula (10) holds for all values of x in the interval $(-\infty, +\infty)$. This formula can also be regarded as the superposition of two such formulae:

$$F(z) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{+\infty} f(u) e^{izu}\, du, \quad f(x) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{+\infty} F(z) e^{-ixz}\, dz\dagger. \quad (13)$$

The function $F(z)$, compared with the first formula for the function $f(x)$, is called its Fourier transform. In its turn the second formula for the function $f(x)$ *is the (inverse) Fourier transform* (the difference is in the sign for i) for the function $F(x)$. We note that the function F will, generally speaking, be complex even when f is real; however, it may be that the original function f is presupposed complex here.

The equation

$$f(x) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{+\infty} F(z) e^{-ixz}\, dz,$$

where the function $f(x)$ is given, may be regarded as an *integral equation* with respect to the unknown function $F(z)$ situated under the integral sign. The solution of the equation is achieved by means of the formula

$$F(z) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{+\infty} f(u) e^{izu}\, du.$$

Naturally, these equations may change roles.

---

† The last integral, if we make only the same hypotheses with respect to $f(x)$ as we discussed above, is understood in the sense of a "principal value".

Now we turn to formula (11); it is satisfied for all positive values of x, and it can be represented as the superposition of the two—on this occasion real and perfectly symmetric—formulae

$$\left.\begin{aligned} F_c(z) &= \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty f(u)\cos zu\,du, \\ f(x) &= \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty F_c(z)\cos xz\,dz. \end{aligned}\right\} \tag{14}$$

Analogously, formula (12) can also be divided into two:

$$\left.\begin{aligned} F_s(z) &= \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty f(u)\sin zu\,du, \\ f(x) &= \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty F_s(z)\sin xz\,dz. \end{aligned}\right\} \tag{15}$$

The functions $F_c(z)$ and $F_s(z)$ are called, respectively, the Fourier cosine transform or sine transform for the function $f(x)$. As we can see, the function f is obtained from F_c (or F_s) in exactly the same way as $F_c(F_s)$ is obtained from f. In other words, the functions f and $F_c(F_s)$ are mutual cosine (sine) transforms. Cauchy called the pairs of functions f and F_c or f and F_s, respectively, conjugate functions of the first and second type. Here also each of the equations (14) [or (15)] can be considered as an *integral equation*, in which the function outside the integral is given, and the function under the integral is to be found; the solution is given by the other equation.

By comparing the functions F, F_c, and F_s, we can make the following remark In the case of an even function $f(x)$ we have

$$F(z) = F_c(z)$$

(for the values $z<0$ the function $F_c(z)$ is extended in an even way), and in the case of an odd function $f(x)$:

$$F(z) = iF_s(z)$$

(for the values $z<0$ the function $F_s(z)$ is extended in an odd way). In the general case the function $f(x)$ can be split up into the sum of an even and an odd function:

$$g(x) = \frac{f(x)+f(-x)}{2}, \qquad h(x) = \frac{f(x)-f(-x)}{2}.$$

Then

$$F(z) = G_c(z) + iH_s(z)\dagger.$$

---

† $G_c(z)$ denotes the cosine transform for the function $g(x)$, and $H_s(z)$ the sine transform for the function $h(x)$.

Because of this it is sufficient to limit ourselves to the examples of cosine and sine transforms.

(1) Let the function be $f(x) = e^{-ax}(a > 0, x \geqslant 0)$; then its cosine transform will be the function

$$F_c(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty e^{-az} \cos zx \, dx = \sqrt{\left(\frac{2}{\pi}\right)} \frac{a}{a^2 + x^2},$$

and the sine transform will be the function

$$F_s(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty e^{-az} \sin zx \, dx = \sqrt{\left(\frac{2}{\pi}\right)} \frac{x}{a^2 + x^2}.$$

Because e^{-ax} is integrable in the interval $[0, \infty]$, the mutual relations must hold:

$$\frac{2a}{\pi} \int_0^\infty \frac{\cos zx}{a^2 + z^2} dz = e^{-ax} \quad (x \geqslant 0),$$

$$\frac{2}{\pi} \int_0^\infty \frac{z \sin zx}{a^2 + z^2} dz = e^{-ax} \quad (x > 0)$$

or

$$\int_0^\infty \frac{\cos xz}{a^2 + z^2} dz = \frac{\pi}{2a} e^{-ax}, \quad \int_0^\infty \frac{z \sin xz}{a^2 + z^2} dz = \frac{\pi}{2} e^{-ax}.$$

We recognize in these integrals *Laplace's well-known integrals* [Sec. 308, *(4)*].

Thus, the pair of functions

$$e^{-ax}, \sqrt{\left(\frac{2}{\pi}\right)} \frac{a}{a^2 + z^2} \quad \text{and} \quad e^{-ax}, \sqrt{\left(\frac{2}{\pi}\right)} \frac{x}{a^2 + x^2}$$

are examples of joint functions of the first and second types (according to Cauchy). If the Laplace integrals were not known to us, the theory expounded would provide a way to calculate them.

(2) Now we consider the function defined by the equations

$$f(x) = \begin{cases} 1 & \text{for} \quad 0 \leqslant x < a \\ \frac{1}{2} & \text{for} \quad x = a \\ 0 & \text{for} \quad x > a. \end{cases} \qquad (a > 0)$$

In this case

$$F_c(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^a \cos zx \, dz = \sqrt{\left(\frac{2}{\pi}\right)} \frac{\sin ax}{x}.$$

Wishing to verify Fourier's formula in this example, we find the cosine transform for the function obtained:

$$\frac{2}{\pi}\int_0^\infty \frac{\sin az}{z}\cos zx\,dz.$$

If we rewrite this integral in the form

$$\frac{1}{\pi}\int_0^\infty \frac{\sin(a+x)z}{z}\,dz+\frac{1}{\pi}\int_0^\infty \frac{\sin(a-x)z}{z}\,dz,$$

then we see immediately [Sec. 293, Remark] that its value, in fact, coincides with the original function $f(x)$.

§ 4. The closed and complete nature of a trigonometrical system of functions

413. Mean approximation to functions. Exstreme properties of segments of a Fourier series. If for any function $f(x)$ we consider another function $g(x)$ as an approximation to it in the interval $[a, b]$, then the *precision* of this approximation can be estimated in various ways. Fundamentally, of course, we consider the difference

$$r(x) = f(x) - g(x).$$

In the cases when we are equally interested in the small deviation of one of these functions from the other at all the points taken separately, as a measure of their proximity we can take their *maximal deviation*

$$\delta = \sup_{a\leqslant x\leqslant b} |r(x)|.$$

However, it is often preferable, instead of the "uniform" proximity of functions at all points, to consider only their proximity "in the mean". In this case, as a measure of their proximity we take their *mean deviation*

$$\delta' = \frac{1}{b-a}\int_a^b |r(x)|\,dx$$

or, which is what we will keep to in the following, *the mean square deviation*

$$\delta'' = \sqrt{\left(\frac{1}{b-a}\int_a^b r^2(x)\,dx\right)}.$$

Instead of this expression, however, it is more convenient to consider the simpler quantity:

$$\Delta = \int_a^b r^2(x)\,dx = (b-a)\delta''^2.$$

We again return to the consideration of an arbitrary system of functions $\{\varphi_m(x)\}$ $(m = 0, 1, 2, \ldots)$, orthogonal in the interval $[a, b]$ [Sec. 398]. Let $f(x)$ be a function given in the same interval† and n a fixed natural number. We consider the following problem: *from all the linear combinations of the first $n+1$ functions φ:*

$$\sigma_n(x) = C_0\varphi_0(x) + C_1\varphi_1(x) + \ldots + C_n\varphi_n(x) \tag{1}$$

with an arbitrary choice of the coefficients $C_0, C_1, \ldots, C_n$, to find the one which realizes the very closest—in the sense of the mean square deviation—approximation to the function $f(x)$. In other words, it is required to obtain the smallest value for the quantity

$$\Delta_n = \int_a^b [f(x) - \sigma_n(x)]^2\,dx.$$

Again substituting for $\sigma_n(x)$ its expanded expression, we obtain

$$\Delta_n = \int_a^b f^2(x)\,dx - 2\sum_{m=0}^{n} C_m \int_a^b f(x)\varphi_m(x)\,dx + \sum_{m=0}^{n} C_m^2 \int_a^b \varphi_m^2(x)\,dx + 2\sum_{k<m} C_k C_m \int_a^b \varphi_k(x)\varphi_m(x)\,dx.$$

The last sum vanishes in view of the orthogonality of our system. Introducing the constants

$$\lambda_m = \int_a^b \varphi_m^2(x)\,dx$$

and the (generalized) Fourier coefficients of the function $f(x)$

$$c_m = \frac{1}{\lambda_m}\int_a^b f(x)\varphi_m(x)\,dx,$$

† As usual, we will consider all the functions under consideration, and φ_m and f, to be continuous or, more generally, piece-wise continuous.

we can rewrite the expression for Δ_n in the form

$$\Delta_n = \int_a^b f^2(x)\,dx - 2\sum_{m=0}^{n} \lambda_m c_m C_m + \sum_{m=0}^{n} \lambda_m C_m^2.$$

In order to obtain a complete square under the summation sign, we need to introduce here the terms $\lambda_m c_m^2$. Providing them with plus and minus signs, we finally obtain:

$$\Delta_n = \int_a^b f^2(x)\,dx - \sum_{m=0}^{n} \lambda_m c_m^2 + \sum_{m=0}^{n} \lambda_m (C_m - c_m)^2.$$

Now it is clear that Δ_n attains its smallest value when the last sum tends to zero, and this will be when

$$C_0 = c_0,\ C_1 = c_1,\ \ldots,\ C_n = c_n.$$

Thus, *of all the polynomials of the form* (1) *it is precisely the segment of the (generalized) Fourier series*

$$s_n(x) = c_0\varphi_0(x) + c_1\varphi_1(x) + \ldots + c_n\varphi_n(x)$$

which provides the quantity Δ_n *with its smallest possible value*

$$\delta_n = \int_a^b [f(x) - s_n(x)]^2\,dx = \int_a^b f^2(x)\,dx - \sum_{m=0}^{n} \lambda_m c_m^2. \tag{2}$$

Again our attention is drawn towards the Fourier coefficients as, in a sense, the "best" of all possibilities. It is important to note here that the coefficients which are shown to be "best" for a fixed n also serve the same purpose for greater values of n, only with the new coefficients adjoined to them.

Equation (2) is called *Bessel's identity*. From it we obtain the inequalities

$$\sum_{m=0}^{n} \lambda_m c_m^2 \leqslant \int_a^b f^2(x)\,dx$$

and (if we take limits as $n \to \infty$)

$$\sum_{m=0}^{\infty} \lambda_m c_m^2 \leqslant \int_a^b f^2(x)\,dx. \tag{3}$$

This is *Bessel's inequality*. It is interesting that the series in (3) is always convergent, whatever may be the function $f(x)$ of the class considered.

As n increases the quantity δ_n decreases, in so far as in its expressions (2) the new negative summands are increased. The larger n is, the nearer the sum $s_n(x)$ "in the mean" approaches the function $f(x)$ under consideration. The question naturally arises: as n increases, is it possible to make the mean square deviation as small as desired, i.e. *does* δ_n *tend to* 0 *as* $n \to \infty$?

If this condition is satisfied, then we say that the sum $s_n(x)$ converges to the function $f(x)$ "in the mean" (which, and we emphasize this, in no way presupposes the "point" convergence of $s_n(x)$ to $f(x)$ in the usual sense of the word). From Bessel's identity it is clear that *the convergence in the mean of the sum* $s_n(x)$ *to the function* $f(x)$ *is equivalent to the existence of the equation* [cf. (3)]

$$\sum_{m=0}^{\infty} \lambda_m c_m^2 = \int_a^b f^2(x)\,dx. \tag{4}$$

This equation is called the *equation of closure*. If it is satisfied, for every continuous function $f(x)$, then the system $\{\varphi_n(x)\}$ is itself called *closed*.

414. The closure of a trigonometrical system. Now we apply all that we have said, in particular, to the trigonometrical system of functions

$$1,\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x,\ \ldots,\ \cos mx,\ \sin mx,\ \ldots \tag{5}$$

in the interval $[0, 2\pi]$†.

Instead of a sum of the form (1) it will be necessary here to consider the trigonometrical polynomials

$$T_n(x) = A_0 + \sum_{m=1}^{n} (A_m \cos mx + B_m \sin mx). \tag{1a}$$

The approximation to the function $f(x)$ "in the mean" realized with their help is characterized by the quantity

$$\Delta_n = \int_0^{2\pi} [f(x) - T_n(x)]^2\,dx.$$

† It is more convenient for us in what follows to use as a basis the interval $[0, 2\pi]$.

As follows from the general argument, *for a fixed n the smallest value of this quantity is that corresponding to the segment of the Fourier series*

$$s_n(x) = \frac{a_0}{2} + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx).$$

The smallest value itself is expressed by the equation

$$\delta_n = \int_0^{2\pi} [f(x) - s_n(x)]^2 dx$$

$$= \int_0^{2\pi} f^2(x) dx - \pi \left\{ \frac{a_0^2}{2} + \sum_{m=1}^{n} (a_m^2 + b_m^2) \right\}, \tag{2a}$$

which represents *Bessel's identity* for the system (5). From it, as in the general case, follows the convergence of the series consisting of the squares of the Fourier coefficients, and *Bessel's inequality*

$$\frac{a_0^2}{2} + \sum_{m=1}^{n} (a_m^2 + b_m^2) \leqslant \frac{1}{\pi} \int_0^{2\pi} f^2(x) dx. \tag{3a}$$

But for the actual system of trigonometrical functions under consideration this result can be made more precise, and the problem raised at the end of the preceding section is completely solved. In fact, the following important theorem holds.

THEOREM. *For any piece-wise continuous function $f(x)$ in the interval $[0, 2\pi]$ it is true that*

$$\lim \delta_n = \lim \int_0^{2\pi} [f(x) - s_n(x)]^2 dx = 0,$$

so that the sum $s_n(x)$ converges to the function $f(x)$ "in the mean".

In other words: *for the given function $f(x)$ the equation of closure*

$$\frac{a_0^2}{2} + \sum_{m=1}^{\infty} (a_m^2 + b_m^2) = \frac{1}{\pi} \int_0^{2\pi} f^2(x) dx^{\dagger} \tag{4a}$$

is always satisfied.

† Although the examination of problems resembling this relation were undertaken at the beginning of the nineteenth century, yet the first rigorous proof of it, apparently, was only given in 1892 by de la Vallée Poussin. One who was

For the proof we will use both of the equivalent formulations, depending on which is convenient. We will carry out the proof in several stages.

(1) We will first establish one auxiliary inequality. We consider two piece-wise continuous functions $f(x)$, $\bar{f}(x)$, and we let $f(x) = \bar{f}(x) \pm \bar{\bar{f}}(x)$. Denoting by dashes above them the quantities corresponding to the functions f and f, obviously, we will have

$$f - s_n = (\bar{f} - \bar{s}_n) \pm (\bar{\bar{f}} - \bar{\bar{s}}_n).$$

From the elementary inequality $(a \pm b)^2 \leqslant 2(a^2 + b^2)$ it follows that

$$(f - s_n)^2 \leqslant 2[(\bar{f} - \bar{s}_n)^2 + (\bar{\bar{f}} - \bar{\bar{s}}_n)^2],$$

and hence

$$\int_0^{2\pi} (f - s_n)^2 dx \leqslant 2\left\{\int_0^{2\pi} (\bar{f} - \bar{s}_n)^2 dx + \int_0^{2\pi} (\bar{\bar{f}} - \bar{\bar{s}}_n)^2 dx\right\}. \tag{6}$$

If as $n \to \infty$ both the integrals on the right tend to 0, then the integral on the left will also tend to 0. In other words: *if the equation of closure is satisfied separately for the two functions $\bar{f}$ and $\bar{\bar{f}}$, then it is also satisfied for their sum or difference $f = \bar{f} + \bar{\bar{f}}$.* This result, obviously, will also be valid for the sum of several functions.

(2) Now we consider the simple function $f_\alpha(x)$, which takes only two values:

$$f_\alpha(x) = \begin{cases} 1, & \text{if } 0 \leqslant x \leqslant \alpha \\ 0 & \text{at all other points.} \end{cases} \quad (0 < \alpha \leqslant 2\pi)$$

For this it is easy to verify immediately that the equation of closure is satisfied by using the expansion which is well known to us [see Sec. 406, *(4)* and Remark]:

$$\frac{x^2}{4} - \frac{\pi x}{2} = \sum_{n=1}^{\infty} \frac{\cos nx - 1}{n^2}.$$

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particularly concerned with the general theory of the closure of an orthogonal system of functions and its applications was the Russian mathematician and academician Vladimir Andreyevich Steklov (1863–1926). The term "equation of closure" is due to him.

In fact, for our function

$$a_0 = \frac{1}{\pi}\int_0^{\alpha} dx = \frac{\alpha}{\pi};\ a_n = \frac{1}{\pi}\int_0^{\alpha} \cos nx = \frac{1}{\pi}\frac{\sin n\alpha}{n};$$

$$b_n = \frac{1}{\pi}\frac{1-\cos n\alpha}{n},$$

so that

$$\frac{a_0^2}{2} + \sum_1^{\infty}(a_n^2 + b_n^2) = \frac{\alpha^2}{2\pi^2} + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin^2 n\alpha + (1-\cos n\alpha)^2}{n^2}$$

$$= \frac{\alpha^2}{2\pi^2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1-\cos n\alpha}{n^2} = \frac{\alpha^2}{2\pi^2} + \frac{2}{\pi^2}\left[\frac{\pi\alpha}{2} - \frac{\alpha^2}{4}\right]$$

$$= \frac{\alpha}{\pi} = \frac{1}{\pi}\int_0^{2\pi}[f_\alpha(x)]^2 dx.$$

Clearly, formula (4a) is true also in the case when the value 1 is replaced by any other constant value c.

(3) Let $f_{\alpha\beta}(x) = f_\beta(x) - f_\alpha(x)$ $(0 < \alpha < \beta \leqslant 2\pi)$; because the equation of closure is satisfied for the functions f_α and f_β, it is also satisfied, by virtue of *(1)* for their difference $f_{\alpha\beta}$, which takes the value 1 inside the interval $[\alpha, \beta]$ and the value 0 outside it; here again the value 1 can be replaced by any constant c.

Now we consider the arbitrary *piece-wise constant* function $\bar{f}(x)$†. Since it can be represented—apart from the values at a finite number of points—in the form of a sum of several functions precisely of the type considered by *(1)*, equation (4a) is valid for it also.

(4) Considering, finally, a piece-wise continuous function, we can again—apart from the values at a finite number of points—regard it as the sum of functions $f(x)$, each of which is continuous in some interval $[\alpha, \beta]$, where $0 \leqslant \alpha < \beta \leqslant 2\pi$, and is equal to 0 outside it. By virtue of *(1)*, for the proof of the theorem it is sufficient to establish the validity of the equation of closure for any function of this type.

† This means that the fundamental interval is divided up into a finite number of parts, inside each of which the function has a constant value.

As a preliminary we transform some inequality (6). In fact, from Bessel's identity (2a)—if we apply it to the function $\bar{\bar{f}}$—it is clear that

$$\int_0^{2\pi} [\bar{\bar{f}}(x) - \bar{\bar{s}}_n(x)]^2 dx \leqslant \int_0^{2\pi} [\bar{\bar{f}}(x)]^2 dx.$$

Therefore the inequality is only strengthened if we write it in the form

$$\int_0^{2\pi} [f(x) - s_n(x)]^2 dx \leqslant 2\left\{\int_0^{2\pi} [\bar{f}(x) - \bar{s}_n(x)]^2 + \int_0^{2\pi} [\bar{\bar{f}}(x)]^2 dx\right\}; \qquad (7)$$

here we assume from the start that $f = \bar{f} + \bar{\bar{f}}$.

Returning to the function f, which we discussed above, we show that for it we can construct a piece-wise constant function $\bar{f}$ such that in the whole fundamental interval $[0, 2\pi]$ it will be true that

$$|f(x) - \bar{f}(x)| < \sqrt{\frac{\varepsilon}{8\pi}}. \qquad (8)$$

In fact, by the corollary to Cantor's theorem [Sec. 75], the interval $[\alpha, \beta]$ can be divided into a finite number of parts $[\alpha_i, \beta_i]$ $(i = 1, 2, \ldots, n)$ so that in each of them the oscillation of the function will be less than $\sqrt{(\varepsilon/8\pi)}$. Now setting the function $\bar{f}$ equal to $f(\alpha_i)$ in each of the intervals $[\alpha_i, \beta_i)$ for $i < n$, and $f(\alpha_n)$ in the interval $[\alpha_n, \beta_n]$, and outside the interval $[\alpha, \beta)$ we let $\bar{f} = 0$. Clearly, the function $\bar{f}$ will be the one we are looking for.

If we take $\bar{\bar{f}} = f - \bar{f}$, then by virtue of (8)

$$|\bar{\bar{f}}| < \sqrt{\frac{\varepsilon}{8\pi}}, \quad \text{so that} \quad \int_0^{2\pi} [\bar{\bar{f}}(x)]^2 dx < \frac{\varepsilon}{4}.$$

But the first integral in the inequality (7) on the right, in view of *(3)*, tends to 0 as $n \to \infty$ and for sufficiently large values of n can also be made less than $\varepsilon/4$. For the same values

$$\int_0^{2\pi} [f(x) - s_n(x)]^2 dx < \varepsilon,$$

which completes the proof.

In particular, it follows from this that *the trigonometrical system of functions* (5) *in the interval* $[0, 2\pi]$ *is closed.*

415. The completeness of a trigonometrical system. The general theorem we have proved has a series of corollaries, which are also of interest.

With its help we can prove the

THEOREM. *Apart from the function which is identically equal to zero there does not exist a continuous function which is orthogonal in the interval* $[0, 2\pi]$† *to all the functions of the trigonometrical system* (5)

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos mx, \sin mx, \ldots.$$

In other words, *if the function* $f(x)$ *continuous in the interval* $[0, 2\pi]$ *has Fourier coefficients all equal to zero, then the function itself reduces identically to zero.*

In fact, according to the equation of closure, for such a function it will be true that

$$\int_0^{2\pi} [f(x)]^2 \, dx = 0.$$

Because the function under the integral here is non-negative, then certainly

$$\int_0^X [f(x)]^2 \, dx = 0$$

for all X in $[0, 2\pi]$. Differentiating with respect to the upper limit, and taking into account the continuity of the function under the integral, we obtain [Sec. 183] that identically

$$f(x) \equiv 0.$$

The property of a trigonometrical system stated in the theorem is called its *completeness*: we say that *the trigonometrical system is complete—in the class of continuous functions.*

Further, *if two continuous functions* $f_1(x)$ *and* $f_2(x)$ *have exactly the same Fourier coefficients, then they are necessarily identical,* because their difference $f_2(x) - f_1(x)$ will have Fourier coefficients all without exception equal to zero. Thus, *a continuous function*

† Or in any other interval of length 2π.

is uniquely defined by its Fourier coefficients. This is only another formulation of the property of completeness of a trigonometrical system.

Remark. All that has been said also holds for any orthogonal system of functions $\{\varphi_m(x)\}$ closed in some interval $[a, b]$: from the equation of closure we obtain the completeness, and so on.

416. The generalized equation of closure. Let there be given two functions $f(x)$ and $g(x)$, continuous or piece-wise continuous in the interval $[0, 2\pi]$; such will also be the functions $f+g$ and $f-g$. If we denote respectively by a_m, b_m and α_m, β_m the Fourier coefficients of the functions f and g, then for the functions $f \pm g$, clearly, the Fourier coefficients will be $a_m \pm \alpha_m, b_m \pm \beta_m$.

Applying the equation of closure separately to the functions $f+g$ and $f-g$, we obtain

$$\frac{(a_0+\alpha_0)^2}{2} + \sum_{m=1}^{\infty} [(a_m+\alpha_m)^2 + (b_m+\beta_m)^2] = \frac{1}{\pi}\int_0^{2\pi} [f+g]^2 dx$$

and

$$\frac{(a_0-\alpha_0)^2}{2} + \sum_{m=1}^{\infty} [(a_m-\alpha_m)^2 + (b_m-\beta_m)^2] = \frac{1}{\pi}\int_0^{2\pi} [f-g]^2 dx.$$

If we subtract these two equations from one another term by term, then, taking into account the identity

$$(a+b)^2 - (a-b)^2 = 4ab,$$

we arrive at the *generalized equation of closure*

$$\frac{a_0\alpha_0}{2} + \sum_{m=1}^{\infty} (a_m\alpha_m + b_m\beta_m) = \frac{1}{\pi}\int_0^{2\pi} f(x)\, g(x)\, dx. \tag{9}$$

Equation (4a) is obtained from this when $g \equiv f$.

Remark. In the general case also of an arbitrary system of functions $\{\varphi_m(x)\}$ orthogonal in the interval $[a, b]$, if the equation of closure (4) is satisfied for every continuous (or piece-wise continuous) function, then for two such functions the generalized equation of closure also holds

$$\sum_{m=0}^{\infty} \lambda_m c_m \gamma_m = \int_a^b f(x)\, g(x)\, dx,$$

if c_m and γ_m are the generalized Fourier coefficients of the functions f and g. This is proved in exactly the same way as the proof just completed.

417. Termwise integration of a Fourier series. We presuppose the function $f(x)$ is, as usual, piece-wise continuous in the interval $[0, 2\pi]$, and we let

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos mx + b_m \sin mx \tag{10}$$

be its Fourier series.

Together with this we also consider the function $g(x)$ (also piece-wise continuous), which is defined by

$$g(x) = \begin{cases} 1, & \text{if } 0 \leqslant x \leqslant x_0, \\ 0 & \text{at other points.} \end{cases}$$

Its Fourier coefficients are equal to

$$\alpha_0 = \frac{1}{\pi} \int_0^{x_0} dx, \quad \alpha_m = \frac{1}{\pi} \int_0^{x_0} \cos mx \, dx,$$

$$\beta_m = \frac{1}{\pi} \int_0^{x_0} \sin mx \, dx \quad (m = 1, 2, 3 \ldots).$$

If we apply the generalized equation of closure (9) to these functions, then we obtain:

$$\int_0^{x_0} \frac{a_0}{2} dx + \sum_{m=1}^{\infty} \int_0^{x_0} (a_m \cos mx + b_m \sin mx) dx = \int_0^{x_0} f(x) dx. \tag{11}$$

Thus, *the integral of the function $f(x)$ is obtained by the termwise* integration of its corresponding Fourier series. The fact that the termwise integration of a Fourier series is always *admissible* is the more remarkable since we established it without even the hypothesis that the series (10) converges to the function $f(x)$†.

† We find the first attempt to prove the admissibility of the termwise integration of a Fourier series in the work of Lobachtevsky (1834).

We can say even more. By substituting in the formula (9), instead of the Fourier coefficients α_m, β_m of the function g, their well-known expressions

$$\alpha_m = \frac{1}{\pi}\int_0^{2\pi} g(x)\cos mx\,dx, \qquad \beta_m = \frac{1}{\pi}\int_0^{2\pi} g(x)\sin mx\,dx,$$
$$(m = 0, 1, 2, \ldots) \qquad\qquad (m = 1, 2, 3, \ldots)$$

we arrive at the equivalent form:

$$\int_0^{2\pi} \frac{a_0}{2} g(x)\,dx + \sum_{m=1}^{\infty} \int_0^{2\pi} (a_m \cos mx + b_m \sin mx)\, g(x)\,dx = \int_0^{2\pi} f(x)\, g(x)\,dx.$$

This equation can be interpreted thus: *the Fourier series* (10) *of the function* $f(x)$ *can be integrated termwise even after its terms have been multiplied by the function* $g(x)$—as the sum we obtain the integral of the product of the functions f and g. Here we are discussing integration in the interval $[0, 2\pi]$. If we are interested in any other interval $[0, x_0]$ $(0 < x_0 < 2\pi)$, then we need only set $g(x) = 0$ outside this interval.

418. The geometrical interpretation. We wish to introduce the reader to a completely new point of view, which we study in this section. We limit ourselves to the examination of continuous functions $f(x)$ in some finite interval $[a, b]$ and we take as our basis an arbitrary closed [Sec. 413] orthogonal system of (continuous) functions $\{\varphi_m(x)\}$; for simplicity we further presuppose this system normal, so that

$$\int_a^b [\varphi_m(x)]^2\,dx = 1 \qquad (x = 0, 1, 2, \ldots). \tag{12}$$

For instance, the reader can assume that we are discussing the normal trigonometrical system

$$\frac{1}{\sqrt{(2\pi)}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \ldots, \frac{\cos mx}{\sqrt{\pi}}, \frac{\sin mx}{\sqrt{\pi}}, \ldots$$

in the interval $[0, 2\pi]$ or $[-\pi, \pi]$ [see Sec. 398, (17*)].

We will consider all functions $f(x)$ as *elements* $\vec{f}$ *of some "vector space"*, defining the addition of two such elements and the multiplication of an element by a scalar c in a natural way:

$$\vec{f}+\vec{g}=f(x)+g(x),\ c\vec{f}=cf(x).$$

By the *length* of a "vector" $\vec{f}=f(x)$, more often called its *norm*, we will understand the non-negative number

$$\|\vec{f}\| = \sqrt{\left\{\int_a^b [f(x)]^2 dx\right\}}. \tag{13}$$

This length is equal to 0, clearly, only for a function $f(x)$ identically equal to 0; the corresponding "vector" is the "zero vector" of our space.

In order to justify the name "length", we need to ascertain that the so-called "triangle axiom" [cf. Sec. 125, (2)]

$$\|\vec{f}+\vec{g}\| \leqslant \|\vec{f}\| + \|\vec{g}\|,$$

is satisfied, i.e. the inequality

$$\sqrt{\left\{\int_a^b [f(x)+g(x)]^2 dx\right\}} \leqslant \sqrt{\left\{\int_a^b [f(x)]^2 dx\right\}} + \sqrt{\left\{\int_a^b [g(x)]^2 dx\right\}}.$$

If we square both sides of this, then it is reduced to the inequality

$$\int_a^b f(x)g(x)\,dx \leqslant \sqrt{\left\{\int_a^b [f(x)]^2 dx\right\}} \sqrt{\left\{\int_a^b [g(x)]^2 dx\right\}} \tag{14}$$

or

$$\left\{\int_a^b f(x)g(x)\,dx\right\}^2 \leqslant \int_a^b [f(x)]^2 dx \int_a^b [g(x)]^2 dx.$$

This is known as *Bunyakovsky's inequality*†, which is an integral

† The academician Victor Yakovlevich Bunyakovsky (1804–1889) was a Russian mathematician. The inequality cited in the text was published by him in 1859. This inequality is often mistakenly connected with Schwarz's name, in whose work it only appeared in 1884.

analogue of Cauchy's algebraic inequality, which we have already met:

$$\sum_{i=1}^{m} a_i b_i \leqslant \sqrt{\left\{\sum_{i=1}^{m} a_i^2\right\}} \sqrt{\left\{\sum_{i=1}^{m} b_i^2\right\}}$$

(see Sec. 125 and the footnote on p. 235 of Vol. I). Bunyakovsky's inequality can be proved in exactly the same way as Cauchy's inequality: the quadratic trinomial with respect to the variable z

$$\int_a^b [zf(x)+g(x)]^2 dx$$

$$z^2 = \int_a^b [f(x)]^2 dx + 2z \int_a^b f(x)\, g(x)\, dx + \int_a^b [g(x)]^2 dx$$

does not take negative values, hence,

$$\int_a^b [f(x)]^2 dx \int_a^b [g(x)]^2 dx - \left\{\int_a^b f(x) g(x)\, dx\right\}^2 \geqslant 0,$$

and so on.

It is convenient to introduce the concept of the "angle" θ between the "vectors" $\vec{f}$ and $\vec{g}$: $\theta = \sphericalangle(\vec{f}, \vec{g})$, defining it by the equation

$$\cos\theta = \frac{\int_a^b f(x) g(x)\, dx}{\sqrt{\left\{\int_a^b [f(x)]^2 dx\right\}} \sqrt{\left\{\int_a^b [g(x)]^2 dx\right\}}}; \qquad (15)$$

the inequality (14) shows that the absolute value of the expression on the right does not exceed unity. The "orthogonality" of the functions $f(x)$ and $g(x)$ [Sec. 398]:

$$\int_a^b f(x)\, g(x) dx = 0,$$

is equivalent to the fact that $\cos\theta = 0$, so that the angle θ is a right angle.

Taking into account the definition of a norm [see (13)], we obtain from (15)

$$\int_a^b f(x) g(x) dx = \|\vec{f}\| \cdot \|\vec{g}\| \cdot \cos\theta;$$

this gives us a basis for considering the integral on the left as the *scalar product* of the "vectors" $\vec{f}$ and $\vec{g}$:

$$\int_a^b f(x)g(x)\,dx = \vec{f}\cdot\vec{g}.$$

If the vector $\vec{g}$ is a "unit vector" or "ort", i.e. if its norm $||\vec{g}|| = 1$, then the scalar product

$$\vec{f}\cdot\vec{g} = ||\vec{f}||\cdot\cos\theta$$

can be regarded as *the projection of the "vector" $\vec{f}$ on this "ort"*:

$$\lim_{\vec{g}} \vec{f}.$$

If there is given a sequence of "vectors" $\{\vec{f}_n\}$, then its *limit* is what we call the "vector" $\vec{f}$, for which

$$||\vec{f}_n - \vec{f}|| \to 0 \quad \text{as} \quad n \to \infty.$$

Returning to the corresponding functions $\{f_n(x)\}$, $f(x)$, we have, by the definition of a norm [see (13)],

$$\int_a^b [f_n(x) - f(x)]^2\,dx \to 0,$$

so that *the convergence of the sequence of "vectors" $\{\vec{f}_n\}$ to a limit "vector" $\vec{f}$ means nothing other than the convergence in the mean [Sec. 413] of the sequence of functions $\{f_n(x)\}$ to the limit function $f(x)$.*

Now we return to the orthogonal normal [see (12)] system of functions $\{\varphi_m(x)\}$ chosen as our basis. It can be treated as a system of mutually orthogonal "orts" $\{\vec{\varphi}_m\}$. In view of the presupposed closure of the given system of functions, it will also be complete [Sec. 415, Remark], and this means that *it is impossible to adjoin any new "ort" to the fundamental system of "orts" $\{\vec{\varphi}_m\}$ which is orthogonal to all the $\vec{\varphi}_m$.*

We consider the projection of an arbitrary "vector" $\vec{f}$ onto any of the fundamental "orts" $\vec{\varphi}_m$:

$$c_m = \lim_{\vec{\varphi}_m} \vec{f} = \vec{f} \cdot \vec{\varphi}_m = \int_a^b f(x)\varphi_m(x)\,dx.$$

We recognize in this expression simply one of the (generalized) Fourier coefficients [Sec. 398] of the function $f(x)$†. In so far as the function $f(x)$ is uniquely determined by its Fourier coefficients [Sec. 415, Remark], *the "vector" $\vec{f}$ is uniquely determined by its projections onto the "orts" of the fundamental system $\{\vec{\varphi}_m\}$.*

The product $c_m\varphi_m(x)$, i.e. the vector $c_m\vec{\varphi}_m = \vec{f}_m$, is naturally called *the component of the "vector" $\vec{f}$ in the direction of the "ort" $\vec{\varphi}_m$.*

Now how can we interpret the generalized Fourier series of the function $f(x)$ [Sec. 398]:

$$c_0\varphi_0(x) + c_1\varphi_1(x) + \ldots + c_m\varphi_m(x) + \ldots ? \tag{16}$$

We note, first of all, that—in view of the presupposed closure of the system $\{\varphi_m(x)\}$, the partial sum $s_n(x)$ of the series (16) converges in the mean to the function $f(x)$ as $n \to \infty$ [Sec. 413], and this means that *the expansion in Fourier series always holds, if we understand convergence to be in the mean.* In other words, the "vector" $\vec{s}_n$ tends to the limit "vector" $\vec{f}$, which can be written in the form of the expansion

$$\vec{f} = c_0\vec{\varphi}_0 + c_1\vec{\varphi}_1 + \ldots + c_m\vec{\varphi}_m + \ldots = \vec{f}_0 + \vec{f}_1 + \ldots + \vec{f}_m + \ldots$$

—each vector $\vec{f}$ is represented in the form of the sum of all its components in the directions of the fundamental "orts". This is the geometrical meaning of the Fourier series.

The equation of closure [Sec. 413], in view of (12), can be rewritten thus

$$\int_a^b [f(x)]^2\,dx = \sum_{m=0}^{\infty} c_m^2$$

or

$$||\vec{f}||^2 = \sum_{m=0}^{\infty} ||\vec{f}_m||^2.$$

† We recall that the functions $\{\varphi_m(x)\}$ are normable, so that all the numbers $\lambda_m = 1$ [see (12)].

The square of the length of the "vector" is equal to the sum of the squares of the lengths of all its components. This is a natural generalization of Piphagor's theorem (corresponding to the case of a two-dimensional vector, expanded into two mutually perpendicular components).

We also recall the generalized equation of closure [Sec. 416, Remark]

$$\int_a^b f(x)g(x)\,dx = \sum_{m=0}^{\infty} c_m\gamma_m,$$

where the c_m are the Fourier coefficients of the function f and the γ_m are the Fourier coefficients of the function g. If we rewrite it thus:

$$\vec{f}\cdot\vec{g} = \sum_{m=0}^{\infty} c_m\vec{\gamma}_m,$$

then here we discern an analogy with the usual expression for a scalar product of two vectors in three-dimensional space in terms of their projections on three mutually perpendicular axes [Sec. 388, (1)].

For a strengthening of the analogy between the "space" functions under consideration and ordinary Euclidean space it would be necessary, first, to extend the class of functions considered and, second, to generalize the definition of an integral which we use. This and other matters will be dealt with later in a course on the so-called theory of functions of a real variable [see Conclusion, part VI].

Remark. Parallel with the vector interpretation of the set $\{f\}$ of all continuous functions, we could consider its point interpretation, where we regard the separate continuous functions $f = f(x)$ as the "points" of the "space" formed by them. In this case it is appropriate to introduce the concept of the *distance* $\varrho\ (f, g)$ between two "points" $f = f(x)$ and $g = g(x)$ in the following way:

$$\varrho(f, g) = \|\vec{f}-\vec{g}\| = \sqrt{\left\{\int_a^b [f(x)-g(x)]^2\,dx\right\}},$$

Then, what we called above the *length* of the "vector" $\vec{f}$ is none other than the *distance* of the "point" f from the "origin" 0.

If f, g, h are three "points" of our "space", then the "triangle" axiom holds in the form

$$\varrho(f, h) \leqslant \varrho(f, g) + \varrho(g, h).$$

This form is obtained from the previous form if we replace $\vec{f}$ by $\vec{f} - \vec{g}$, and $\vec{g}$ by $\vec{g} - \vec{h}$.

§ 5. An outline of the history of trigonometrical series

419. The problem of the vibration of a string. This famous problem, which attracted the attention of a series of outstanding mathematicians in the eighteenth century, played an important part in the actual statement of *the problem of the possibility of a trigonometrical expansion for an arbitrary function.* Therefore we will first explain to the reader the given problem and its solution.

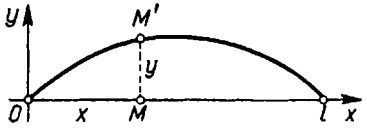

FIG. 81.

Let a homogeneous string, of length l, be fastened at its end-points to the points $x = 0$ and $x = l$ of the x-axis, and under the action of some tension let it lie in equilibrium along this axis (Fig. 81). We suppose that at the moment $t = 0$ the string is displaced from its position of equilibrium in the xy plane (which, for instance, we will presuppose vertical) and then is left to itself†. Then the points of the string begin to oscillate in the vertical plane. If we suppose that every point M of the string with abscissa x oscillates strictly vertically, then its position at the moment of time $t \geqslant 0$ is determined by its ordinate y, which represents its deviation from the position of equilibrium. This ordinate will be a function of both the variables x and t:

$$y = y(x, t), \tag{1}$$

which we need to determine.

Under certain conditions, which simplify the problem, the motion is described by the following differential equation in partial derivatives:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}, \tag{2}$$

where a is a constant, depending on the physical properties of the string.

† In order to be specific we limit ourselves to the hypothesis that the vibration of the string starts from a position of rest. In fact, this hypothesis is the basis of the solution of the problem given by Euler (see the following section).

Besides this equation, the required function $y(x, t)$ must satisfy another series of requirements. First of all, the so-called limit conditions must be satisfied:

$$y(0, t) = 0, \quad y = (l, t) = 0 \quad (t \geqslant 0), \tag{3}$$

which express the fact that the ends of the string are fixed. Then, *if at the first moment* $t = 0$, *the deviations of the points of the line are characterized by the function* $f(x)$ $(0 \leqslant x \leqslant l)$, *and the velocity is equal to zero*†, then it is also necessary to require that the initial conditions

$$y(x, 0) = f(x), \quad \frac{\partial y(x, 0)}{\partial t} = 0 \quad (0 \leqslant x \leqslant l) \tag{4}$$

are satisfied.

Thus, finally, the problem reduces to finding such a function $y(x, t)$, which for $0 \leqslant x \leqslant l$ and $t \geqslant 0$ satisfies equation (2) and conditions (3) and (4).

Remark. In so far as the function $y(x, t)$, *which becomes* $f(x)$ *when* $t = 0$, must satisfy equation (2) and conditions (3), *in particular, at the first moment*, then the function $f(x)$ itself must necessarily be presupposed here to be twice differentiable in the interval $[0, l]$ and, moreover, must satisfy the conditions

$$f(0) = 0, \quad f(l) = 0, \tag{5a}$$

and also

$$f''(0) = 0, \quad f''(l) = 0 \tag{5b}$$

420. D'Alembert's and Euler's solution. These mathematicians were the first to formulate equation (2) clearly. Both D'Alembert (first in 1747) and Euler (later, in 1748) represented its general solution in the form

$$y = \varphi(x + at) + \psi(x - at). \tag{6}$$

Here $\varphi(u)$ and $\psi(v)$ denote "arbitrary" functions, given for all values of the arguments. For our purpose it is sufficient to verify immediately that the expression (6) satisfies condition (2) for all values of x and t, whatever may be the twice differentiable functions φ and ψ.

Now the question arises of the choice of these functions in such a way that the limit and initial conditions are also satisfied, i.e. so that [see (3)]

$$\varphi(at) + \psi(-at) = 0, \quad \varphi(l + at) + \psi(l - at) = 0 \quad (t \geqslant 0) \tag{3a}$$

and [see (4)]

$$\varphi(x) + \psi(x) = f(x), \quad \varphi'(x) - \psi'(x) = 0 \quad (0 \leqslant x \leqslant l). \tag{4a}$$

Here, clearly, it is sufficient to determine the function $\varphi(u)$ for $u \geqslant 0$, and the function $\psi(v)$ for $v \leqslant l$.

First of all, integrating the second of the equations (4a), we find

$$\varphi(x) = \psi(x)\dagger \quad (0 \leqslant x \leqslant l).$$

† We cannot introduce the constant of integration C without loss of generality, because, if we substract $C/2$ from φ and add $C/2$ to ψ, the given equation can always be realized.

Then, by virtue of the first of these equations, both the functions are equal to $f(x)/2$: thus, in the interval $[0, l]$ they are already defined (their graph in Fig. 82 is represented by the curve OMA). Because the function φ need only be defined in the interval $(l, +\infty)$ and the function ψ in the interval $(-\infty, 0)$ which does not depend on it, then for simplicity of notation we can replace the functional characteristic ψ by φ and define the function $\varphi(x)$ in these intervals, starting from the conditions [see (3a)]

$$\varphi(x)+\varphi(-x)=0, \quad (x \geqslant 0) \tag{7}$$

$$\varphi(l+x)+\varphi(l-x)=0. \tag{8}$$

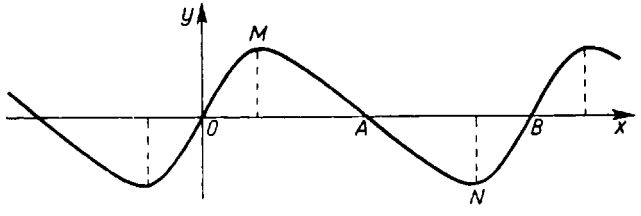

FIG. 82.

First of all, by varying x from 0 to l in (8), we determine φ in the interval $(l, 2l]$:

$$\varphi(l+x)=-\varphi(l-x)=-\tfrac{1}{2}f(l-x)$$

(the curve ANB in Fig. 82). Now, when the function φ is completely determined in the interval $[0, 2l]$, where, by virtue of (5a),

$$\varphi(0)=\varphi(2l)=0,$$

we will prove that in general it has period $2l$, and at the same time we will define it in the whole interval $(-\infty, +\infty)$.

First let $\xi \geqslant -l$; setting $x=\xi+l$ in (8), we obtain [on account of (7)]

$$\varphi(\xi+2l)=-\varphi(-\xi)=\varphi(\xi).$$

Analogously, when $\xi<-l$ we take $x=-(\xi+l)$ in (8) and [again relying on (7)] we arrive at the same result. In general, as the graph of the function φ we obtain (in Euler's expression) the "snakelike curve" illustrated in Fig. 82. It is symmetrical with respect to the origin, as it must be, because by virtue of (7) the function $\varphi(x)$ is odd.

Because in the interval $[0, l]$ the function $f(x)$ has two derivatives and, at the same time, satisfies the conditions (5a) and (5b), then, as we can show, the function $\varphi(x)$ defined in the given way will be twice differentiable everywhere, not excluding points of the form kl ($k=0, \pm 1, \pm 2, \ldots$), where the separate

pieces of the curve are "joined together". For this function the desired solution can finally be represented in the form

$$y = \varphi(x + at) + \varphi(x - at). \tag{9}$$

421. Taylor's and D. Bernoulli's solution†. Already in 1713 Taylor was studying the problem of the motion of a stretched string and he determined the period of its oscillation. As his starting point he took an hypothesis essentially equivalent to equation (2). As a result he arrived at the deduction that, *if the string oscillates as a whole*, then at every moment it has the form of an arc of a sine curve.

Of course, this is only one of the possible forms of oscillation for a string. From Taylor's reasoning follows the existence of an innumerable set of such forms: it is only necessary to divide the string into $n (\geqslant 2)$ equal parts and to apply the previous deduction to each of them separately. In this case the form of the string represents n successive sine arcs.

We obtain all these partial solutions of equation (2) in a precisely analogous way, starting immediately from the equation. We begin to look for its solution (different from the obvious solution, which is of no interest to us—zero) in the form of the product of two functions, of which one depends only on x and the other only on t:

$$y = X(x) \cdot T(t).$$

In this case equation (2) takes the form

$$XT'' = a^2 X'' T$$

(where the dashes denote derivatives with respect to the variable on which the function depends) or, finally,

$$\frac{T''}{T} = a^2 \frac{X''}{X}. \tag{10}$$

Because the left side of this equation does not depend on x, and the right does not depend on t, their common value in fact depends neither on x nor on t, i.e. it reduces to a constant, which we take in the form $-a^2\lambda^2$ (where $\lambda > 0$). Then equation (10) can be divided into two:

$$T'' + a^2\lambda^2 T = 0, \quad X'' + \lambda^2 X = 0;$$

the general solutions of these ordinary, differential equations, respectively, have the form:

$$T = A \cos a\lambda t + B \sin a\lambda t,$$
$$X = C \cos \lambda x + D \sin \lambda x.$$

In order that the function $y = XT$ should also satisfy the limit conditions (3), the function X must satisfy them. Letting $x = 0$ in it, we see at once that $C = 0$;

† Daniel Bernoulli (1700–1782), son of Johann Bernoulli and friend of Euler. In the period from 1725 to 1733 he worked at the St. Petersburg Academy of Sciences.

letting $x = l$ and taking into account that D cannot be zero, we arrive at the condition

$$\sin \lambda l = 0,$$

whence $\lambda l = n\pi$ for natural n. Thus, λ can have only one of the following values:

$$\lambda_1 = \frac{\pi}{l}, \quad \lambda_2 = \frac{2\pi}{l}, \quad \ldots, \quad \lambda_n = \frac{n\pi}{l}, \quad \ldots.^{\dagger}$$

If we assume that the initial velocity of all the points of the string is equal to zero, then $T'(0)$, and hence $B = 0$. When $\lambda = \lambda_n$, denoting the constant AD by b_n, we arrive at the successive partial solutions of equation (2) which we mentioned:

$$y_n = b_n \cos \frac{n\pi a}{l} t \sin \frac{n\pi}{l} x.$$

If the string oscillates according to such a law, then *all its points oscillate with precisely the same period*

$$T_n = \frac{2l}{na},$$

to which corresponds a sound of definite pitch. The amplitude of the oscillation of each point depends on its position and is equal to

$$\left| b_n \sin \frac{n\pi}{l} x \right|.$$

The whole string is split up into n equal portions, where the points of one portion all lie in the same phase and the points of the neighbouring portions in opposite phases. The successive positions of the string, for the cases $n = 1, 2, 3, 4$, are illustrated in Fig. 83. The points separating one portion from another are at rest; these are the so-called "nodes". The centres of the portions—the "peaks"—oscillate with the greatest amplitude.

After the appearance of the general investigations of d'Alembert and Euler on the oscillation of a string, Daniel Bernoulli started to work on the same subject (1753). Recalling that the multiplicity of the forms of oscillation of a string was not unknown to Taylor, Bernoulli—relying on physical observations—made the assertion that *all these oscillations are realized at once, and the string simultaneously produces sounds of different pitches.* There occurs, in his expression, a "mixing" (we would say—a superposition) of the separate oscillations, so that the complete oscillation of the string can be characterized by the equation

$$y = \sum_{n=1}^{\infty} b_n \cos \frac{n\pi a}{l} t \sin \frac{n\pi}{l} x. \tag{11}$$

---

† If we took the constant value of the relation (10) in the form $a^2\lambda^2$, then the limit conditions could only be satisfied by the function X, identically equal to zero.

The fundamental tone is determined by the first component, y_1: it corresponds to the period $T_1 = 2l/a$. The remaining sounds produced by the string, or overtones, are characterized by the other components; they correspond to the periods $T_2 = T_1/2$, $T_3 = T_1/3$, ... If we touch the middle of the string with a finger, then at once we muffle both the fundamental tone and the odd overtones, for which it was a peak. The even overtones, for which the middle of the string was a node, all remain; for them the part of the fundamental tone will be played by the second overtone, and the string produces the octave of the original tone.

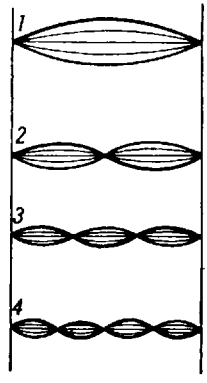

FIG. 83.

In fact, the wealth of physical phenomena, in complete harmony with his experiment, compelled Bernoulli to prefer solution (11) to Euler's solution (9). It is necessary to say that Euler himself mentioned solution (11), but only as a possible partial solution. For Bernoulli in his turn—by his own admission—the problem of the generality of his solution was never sufficiently clear, but he failed to understand what (physical) meaning other solutions might have, if they existed.

We will prove now, relying on the theory proved above, that—*with the proper choice of coefficients in formula* (11)†—*it is identical with formula* (9). Here we presuppose, of course, that the conditions with respect to the function $f(x)$, which were imposed on it earlier, are satisfied [Sec. 419, Remark].

We begin with the choice of the coefficients b_n. If we wish formula (11) to give a solution satisfying the initial conditions (4):

$$y(x, 0) = f(x) \quad (0 \leqslant x \leqslant l),$$

then, if we set $t = 0$ in (11), we must have

$$\sum_1^\infty b_n \sin \frac{n\pi}{l} x = f(x) \quad (0 \leqslant x \leqslant l). \tag{12}$$

† Bernoulli did not touch on the problem of the coefficients.

Because the function $f(x)$ is differentiable and vanishes at the end-points of the interval $[0, l]$, then such an expansion in fact exists in the whole interval, if we take [see Sec. 405, (22)]

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx, \quad (n = 1, 2, 3, \ldots),$$

which we will also presuppose.

But the series (12) converges for all values of x, and from it the function

$$\varphi(x) = \frac{1}{2}\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

can be defined in the whole interval from $-\infty$ to $+\infty$. Thus, in the interval $[0, l]$ this function is equal to $f(x)/2$; moreover, as it is easy to verify immediately, conditions (7) and (8) are satisfied for it. Hence it follows that it coincides precisely with the function φ, which we discussed in the previous section, in so far as all the conditions mentioned, together with those we have taken, determine it uniquely.

In this case Euler's solution (9), which is already known to us, can now be represented in the form

$$y = \frac{1}{2}\sum_{n=1}^{\infty} b_n\left[\sin\frac{n\pi(x+at)}{l} + \sin\frac{n\pi(x-at)}{l}\right] = \sum_{n=1}^{\infty} b_n \cos\frac{n\pi a}{l}\,t \sin\frac{n\pi}{l}\,x,$$

so that the expansion (11) with the given choice of coefficients in fact gives the solution of equation (2), and moreover is identical with the previous one.

422. The controversy concerning the problem of the vibration of a string. There were two sides to this controversy.

First of all, in spite of the formal resemblance of the solutions proposed by d'Alembert and Euler, it soon became clear that there exists a great divergence between the two theories on the question of *which functions it is permissible to consider in analysis*, and, in particular, what the function f must be, which characterizes the initial form of the string.

Euler did not require that this function be defined in the whole interval $[0, l]$ by one analytical expression (Euler called such a function "continuous"). This function may also be "displaced", i.e. it may be given in different parts of the interval by different expressions. Euler even considered the possibility of defining the initial position of the string simply as a "curve, drawn with a free movement of the hand": in fact, in this case also it was possible to realize the construction of the graph of the function φ indicated by us (Fig. 82), and for it, it was easy to establish geometrically the form of the string at any moment of time.

D'Alembert's position was quite different: not only the initial function $f(x)$, but also the function $\varphi(x)$ defined from it in the infinite interval, must, so he

was convinced, be subject to one analytical rule; the solution of the problem by means of analysis is possible only in the case when "the different positions of the oscillating string can be included in precisely one equation". Afterwards d'Alembert concentrated his objections on the problem of a curve which must be defined at all points, in so far as the equation itself contains the second derivative $\partial^2 y/\partial x^2$; in the meantime, the arbitrary way in which f is given and the graph of φ "put together" from the pieces do not guarantee its existence.

If we try to satisfy equation (2), then it is impossible to dismiss the last consideration with assurance [cf. the restrictions which we imposed above on the function f (Sec. 419, Remark)].

The problem of the nature of the "arbitrary functions", which first occurred in the solution of differential equations in partial derivatives†, was debated for a long time amongst mathematicians. In 1787, after the death of both the disputants, the St. Petersburg Academy drew out this controversial question into a competition for a prize, which was presented in 1790 to Louis Arbog, who was close in his ideas to Euler.

Now we go on to the second subject of controversy, which has an immediate connection with the fundamental consideration of the present outline. This time Euler and d'Alembert were in the one camp, and both opposed Bernouilli, asserting that his solution was not general, because it far from guaranteed all the possible rules for the oscillation of a string.

Euler clearly understood that everything reduces to the question: *can any function $f(x)$* (which characterizes the initial form of the string) *be expanded as a series of sines of multiple arcs?* He himself answered this question in the negative. First of all, the initial curve, along which the string lies, may not be expressed at all by any equation, and this means, in particular, by Bernoulli's equation (12). But even if it is represented by an analytical expression, then this latter may not possess the peculiarities which distinguish every curve expressed by an equation of the form (12). Amongst these peculiarities is the fact that, first, the odd ordinates are functions of the abscissae, and, secondly, that this function has period $2l$.

Euler's objection requires an explanation: in fact, the function $f(x)$ is given only in the interval from 0 to l, where these peculiarities in general cannot be shown. But at the time it was considered to be self-evident that *two analytical expressions, which take equal values as the variable varies in some interval, by the same token are also identified everywhere outside it.* Therefore, for instance, if the function $f(x)$ were given in the interval $(0, l)$ by means of an algebraic expression, which never possesses the second of the properties mentioned, then—by virtue of what we have said—even within the limits of this interval alone it cannot coincide with an expression of the form (12).

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† Like the arbitrary constants already encountered in the solution of ordinary differential equations.

D'Alembert in his objections went even further: in his opinion, even if the separate parts of Euler's curve were all connected by one analytical expression, the whole of it need not obey the rule given by Bernoulli. Thus d'Alembert rejected the solution which realized precisely his own requirements, that the different positions of the oscillating string were "included in precisely one equation".

In his turn, Bernoulli now asserted with greater insistence that an equation of the form (12) encompassed all possible curves: by setting out an infinite set of coefficients, we can compel the curve expressed by it to pass through any number of points given in advance. Here, as before, the physical side of the problem was of particular importance in his eyes, and in fact "the possibility of reducing motions existing in nature, which do not appear to be subject to any rule, to simple isochronous motions...".

423. The expansion of functions in trigonometrical series; the determination of coefficients. The striking part in all this discussion was that Euler at this time had given examples of the expansion of algebraic functions in trigonometrical series (of course, obtained completely formally).

In *Differential Calculus* (1755) we find the expansion

$$\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4} + \ldots = \frac{\pi - x}{2} \tag{13}$$

[cf. Sec. 406, *(1)*], which Euler had already communicated by letter in 1744 to his friend Goldbach. In 1760 a work of Euler's was published, which had been written in 1754, where, starting from the expansion

$$\frac{\cos x - a}{1 + a^2 - 2a\cos x} = \cos x + a\cos 2x + a^2\cos 3x + a^3\cos 4x + \ldots$$

and, letting $a = \pm 1$, he first arrived at the divergent series

$$\begin{aligned} \cos x + \cos 2x + \cos 3x + \cos 4x + \ldots &= -\tfrac{1}{2}, \\ \cos x - \cos 2x + \cos 3x - \cos 4x + \ldots &= \tfrac{1}{2}, \end{aligned} \tag{14}$$

and then he integrated the second of these termwise and obtained the result:

$$\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots = \frac{x}{2} \tag{15}$$

[cf. Sec. 406, *(2)*]. "There is no need to add a constant", explained Euler, "because when $x = 0$ the sum itself vanishes." Euler gave no indications as to the domain of applicability of these and similar expansions.

It is interesting that Euler did not obtain the expansion (13) by the termwise integration of the first of the series (14). This was only carried out afterwards by Daniel Bernoulli. He emphasized first of all that the definition of the constant of integration in the equation

$$\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4} + \ldots = C - \frac{x}{2}$$

requires "care": for instance, it is of no use for this purpose simply to substitute $x = 0$; Bernoulli set $x = \pi/2$ and, using the known series of Leibniz [Sec. 255, (20)], arrived at the equation

$$C - \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots = \frac{\pi}{4}, \quad \text{from which} \quad C = \frac{\pi}{2}.$$

Bernoulli (first) indicated exactly the domain of applicability of a termwise expansion—the interval from 0 to 2π, where for $x = 0\ (2\pi)$ the equation does not hold. More than this: for other intervals: from 2π to 4π, from 4π to 6π, and so on it is necessary to vary the constant of integration correspondingly (although this also "seems contrary to the law of continuity").

The important question of *defining the coefficients of the trigonometrical expansion of a function given in advance*—in general form—was apparently first raised by Claireau in 1757 in a work on theoretical astronomy. In it he discussed the expansion of a function in the interval from 0 to π in a series of cosines. First the author solved the interpolation problem, limiting himself to a finite number n of terms of the series and making their sum coincide with the value of the function at the same number of equidistant points; the coefficients are obtained in the form of certain sums. Only then did he "take n to be infinite", i.e. essentially, he took limits, and established the "exact" values of the coefficients in the form of the integral expressions known to us [Sec. 405, (16)]†.

The same formulae were obtained by Euler himself in 1777, this time *by exactly the same method as we now use, i.e. by means of termwise integration.* (Euler's work was published after a considerable delay—in 1798.) Euler formulated his result in the form of a "general theorem", in the conditions of which was contained the hypothesis that the function should admit an expansion in a series of cosine terms. *On the question of whether an arbitrary function admits such an expansion, there is no discussion here.*

A complete change in the actual attitude to this problem was brought about by Fourier in his researches at the beginning of the last century—researches dedicated to the theory of the propagation of heat. These researches, which were communicated to the Paris Academy in the form of separate reports and memoranda, starting in 1807, were to a considerable extent summed up in the work *The Analytical Theory of Heat* (1822) and quickly became well known.

Fourier, like Euler, also established the integral formulae for the coefficients of the trigonometrical expansion of a function by means of termwise integration. Because it is possible by means of these formulae to calculate easily the coefficients for "completely arbitrary functions" he came to the conclusion that all such functions *admit an expansion.* A strong impression was made on Fourier's con-

† The interpolation problem—in connection with the form of the oscillating string—was also solved several years later by Lagrange.

temporaries by the expansions of functions which are subject to different rules in different parts of the interval, for instance:

$$\sum_{n=1}^{\infty} \frac{1-\cos n\alpha}{n} \sin nx = \begin{cases} \pi/2 & \text{for} \quad 0<x<\alpha, \\ 0 & \text{for} \quad \alpha<x<\pi. \end{cases}$$

Fourier expanded (in the interval from 0 to π) the even function $\cos x$ in sine terms, and the odd function $\sin x$ in cosine terms. A decisive blow was dealt to all this by his previous conviction: *in fact it smoothes over the difference between "mixed" and "continuous" functions* (in Euler's terminology). Together with this, *the old controversy on the trigonometrical expansion of an "arbitrary" function was definitely decided in Bernoulli's favour!*

In a series of cases it was possible to verify immediately the convergence of the expansions constructed by Fourier to the corresponding functions. In addition, these expansions all find a very wide application in mathematical physics. However, notwithstanding all this, Fourier's ideas met with little sympathy at first: termwise integration was customary at the time, but in fact it did not prove, but presupposed the admissibility of an expansion, and so it was a hindrance to the proof of this admissibility.

424. The proof of the convergence of Fourier series and other problems. It is true that Fourier himself at the end of his work made an attempt to prove the validity of the trigonometrical expansion of a function, but his reasoning was far from analytically rigorous, although it was based on his idea, which he gave geometrical form, and was correct. Then attempts by other authors followed, amongst whom was Cauchy, but they also were criticized.

The first actual rigorous proof of the assertion made by Fourier was due to Dirichlet (1829); it essentially used Fourier's own idea. Although in the title of the work referred to here† mention is made of "arbitrary functions", in fact Dirichlet limited himself precisely to the consideration of a class of functions: these were the functions defined and finite in the interval $[-\pi, \pi]$, piece-wise continuous and piece-wise monotonic‡. If the function $f(x)$ satisfies these requirements, then—as Dirichlet proved—the trigonometrical series constructed for it by Fourier's formulae converges to the sum

$$\frac{f(x+0)+f(x-0)}{2}$$

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† G. Lejeune Dirichlet, "On the convergence of trigonometrical series, serving as a representation within given limits of an arbitrary function" (Kharkov Mathematical Library, Series C, Vol. 2, 1914, pp. 3–23.

‡ This means that the fundamental interval may be divided up into a finite number of parts, inside each of which separately the function varies monotonically.

where $-\pi < x < \pi$, and to the sum

$$\frac{f(-\pi+0)+f(\pi-0)}{2}$$

where $x = \pm\pi$.

Within a short time (in 1834–1835) Lobatchevsky also gave a proof of the theorem on expansions, imposing, however, different conditions on the function from those of Dirichlet. In fact, Lobatchevsky supposed that the function in general was differentiable, with the exclusion of perhaps just a finite number of jumps, and also the points where the function itself or its derivative tends on one side or the other to $\pm\infty$; here the function must be integrable—in the proper or improper sense†. Dirichlet's and Lobatchevsky's conditions are not interchangeable.

Afterwards other authors established a whole series of more general sufficiency criteria for the convergence of a Fourier series to the original function.

An important place in the development of the theory of trigonometrical series is occupied by a well-known dissertation by Riemann‡ (1854; published in 1867). It begins with an outline of the history of the problem. Then the author pauses to make more precise the concept of a definite integral and he establishes conditions for its existence. In this way he extended the domain of applicability of Fourier series (in particular, for instance, he made it clear that it is not necessary to make any hypothesis about the continuity of the function in Dirichlet's theorem). However, the main part of the work is concerned with the consideration of trigonometrical series of a general form

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx + b_n\sin nx \tag{16}$$

and with clarifying the necessary conditions for a similar series to represent an arbitrary function $f(x)$ with period 2π. Riemann compared the series (16) with the series

$$\frac{a_0}{4}x^2-\sum_{n=1}^{\infty}\frac{a_n\cos nx+b_n\sin nx}{n^2}$$

which he obtained from it by integrating twice formally termwise. Under the hypothesis that $a_n \to 0$ and $b_n \to 0$ (and we can always reduce the problem to this case), the latter series converges uniformly in the whole interval from $-\infty$ to $+\infty$ and defines a continuous function $F(x)$ in it. The last part of the work

† N. I. Lobatchevsky, "On the vanishing of trigonometrical series" and "A method of ascertaining the vanishing of infinite series..." (*The Complete Collected Works*, Vol. V, 1951, pp. 31–80 and 81–162).

‡ Bernhard Riemann, "On the possibility of representing a function by means of a trigonometrical series" (the collection mentioned in the footnote on the previous page, pp. 27–85, or his *Works*, 1948, pp. 225–261).

is devoted to the properties of the function $F(x)$ and its connection with the function $f(x)$. A series of works by other authors followed Riemann's dissertation, using his method of examining the series (16) and the results obtained from it.

First of all there arose *the problem of a unique trigonometrical expansion for a function*: in fact, the Euler–Fourier formulae, giving the coefficients of such an expansion, were established by means of the termwise integration of a series, and in the second half of the past century it was realized that it is not admissible to use such a method without reservations. The problem mentioned was first raised in 1870 by Heine†, but the complete solution to it was then given by Cantor, who proved the following general theorem: *if for the function $f(x)$ in the interval $[-\pi, \pi]$ an expansion in a series of the form* (16) *is in general possible, then such an expansion is unique.* The uniqueness is preserved also when there are in the interval indicated exceptional points, for which nothing is known about the sum of the series or about its convergence (only if the set of such exceptional points is infinite is it subject to certain restrictions).

What are the coefficients of this unique expansion of the function $f(x)$ (if it is admissible)? *Will they always be precisely the Fourier coefficients?* Of course, this problem can only be solved with respect to those functions for which the integrals, figuring in the Euler–Fourier formulae, in general have a meaning. It is very remarkable that *the answer is in the affirmative*, as was first established in 1872 for continuous functions by Ascoli‡, and in 1874 for functions integrable in the proper sense by du Bua-Ramon§. Afterwards this result was also extended to functions which are integrable in a more general sense. *These works first convincingly justified the exceptional attention which was devoted to the study of the expansion of functions in Fourier series.*

We will only stop to consider one further question. Dirichlet was convinced that every continuous function, with period 2π, could be expanded in a Fourier series, although he could not prove it. This assertion, it seems, was shared by other mathematicians. However, du Bua-Ramon, after a series of fruitless attempts to prove the validity of Dirichlet's proposition, refuted it by an example, by constructing a continuous function for which the Fourier series diverges at an innumerable set of points within the limits of as small an interval as desired.

With this we complete our outline, because—although the elementary part of the theory of trigonometrical series certainly relates to the fundamentals of analysis—in its further development this theory is concerned with more subtle concepts and assertions which will be unfamiliar to the reader.

425. Concluding remarks. We would wish to emphasize here the exceptional position which the theory of trigonometrical series occupies in the history of analysis.

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† Heinrich Edward Heine (1821–1881)—a German mathematician.

‡ Gulio Ascoli (1843–1896)—an Italian mathematician.

§ Paul du Bua-Ramon (1831–1889)—a German mathematician, of Swiss descent.

First of all, there is a close connection between this theory and the refinement of the concept of a function. The controversy between Euler and Daniel Bernoulli in connection with the expansion of an "arbitrary" function in a trigonometrical series contributed much to removing certain preconceived and incorrect ideas, was which completed in the works of Fourier. We find the contemporary more general definition of a function, not by accident, in the works of Lobatchevsky and Dirichlet, in fact in their investigations of trigonometrical series.

In connection with the requirements of the theory of Fourier series, Riemann made more precise, and generalized, the concept of a definite integral. Cantor made his first steps in the theory of infinite sets, which was created by him, in connection with the uniqueness of a trigonometrical expansion. The contemporary theory of the generalized summation of divergent series originated in the work of Poisson† on the summation of trigonometrical series. Even if these or other subtle concepts of analysis (for instance, the absolute and non-absolute convergence of a numerical series, the uniform and non-uniform convergence of a functional series) seem unconnected with the theory of trigonometrical series, they are immediately applicable to these series, which serve as a touch-stone to test their importance. This is the state of affairs at the present time.

Trigonometrical series have an immediate application in mathematical physics (the reader can see an illustration of this in the example considered above on the problem of an oscillating string) and in many branches of technology. However, the so-called "generalized Fourier series" have an even wider domain of applicability, i.e. expansions in different systems of orthogonal functions, for which the trigonometrical expansions serve as an example, and the theory of which is closely related to the theory of Fourier series.

† Simon Denis Poisson (1781–1840)—an outstanding French mechanist, physicist and mathematician.

CONCLUSION

AN OUTLINE OF FURTHER DEVELOPMENTS IN MATHEMATICAL ANALYSIS

AFTER having become familiar with the fundamentals of mathematical analysis, the reader may find it interesting to become acquainted, if only in very general terms, with the mathematical disciplines, which on the whole nowadays form part of mathematical analysis, understood in its wider sense.

I. The theory of differential equations

The reader will already be familiar with this offshoot of analysis from a university course, which enables us to be brief in the given case.

Even in the early history of differential and integral calculus there occurred—as they were called then—"converses to the problems of tangency", i.e. problems in which one is trying to find curves when the properties of their tangents are given. Mention is made of these problems in the correspondence between Newton and Leibniz [Sec. 228]; it was in this that Leibniz first used the term "differential equations".

The second fundamental problem was given by Newton in his *Method of Fluxions*: *from given equations containing fluxions, to find the relation between the fluents*, as we indicated in Sec. 225; this is a general problem on the integration of an *ordinary differential equation*. Newton, as a rule, solved it with the aid of power series, without trying to represent the solution in the form of a "finite" analytical expression. Leibniz and the brothers Bernoulli made a considerable study of differential equations, sometimes also using infinite series. However, at the same time the first systematic

attempt to solve certain types of equation of the first order by reduction to a square is due to them.

Only in the eighteenth century was the theory of differential equations developed enough that it came to be regarded as an independent scientific field. A large part in this was played by the numerous and various works of Euler.

We recall that in fact Euler introduced the concepts of a "complete" (general) and "partial" solution of an ordinary differential equation; he also made a considerable study of solutions not contained in the complete integral (singular solutions). He widely developed the method of integrating factors not only for equations of the first, but also for equations of higher orders. Euler originated the method of the solution of a homogeneous linear differential equation with constant coefficients by means of the characteristic equation and clarified the form of its general integral. In connection with the vibration of a circular membrane Euler first began to consider the general equation of cylindrical functions

$$\left(\alpha^2 - \frac{\beta^2}{r^2}\right)u + \frac{1}{r}\frac{du}{dr} + \frac{d^2u}{dr^2} = 0$$

(later called "Bessel's equation"), representing its solution in the form of an infinite series. Euler originated the first method for the approximate solution of ordinary differential equations.

The works of Euler, together with the works of Claireau, Lagrange and other mathematicians of the eighteenth century, further advanced the formal theory of *ordinary differential equations.*

Now we turn to *differential equations in partial derivatives.* Euler has a considerable reputation in this field also. He devoted a series of investigations to equations of the first order. In addition, Euler often reduced many problems in mechanics and physics to the consideration of equations of the second (and higher) orders†. The peculiarity of equations in partial derivatives was the appearance in part of the solution of arbitrary functions. D'Alembert first noticed this in the example of the equation of the oscillation of a string [Sec. 420], but in fact Euler used arbitrary functions in order

† Similar problems and the solution of the corresponding equations form the subject matter of so-called *mathematical physics.*

to satisfy the original data, arising from the problem under consideration, and in this way to complete his investigation.

The various works of Euler's contemporary, Daniel Bernoulli, were of considerable influence in the subsequent development of mathematical physics. He originated a general method for the solution of the problem of the oscillation of an elastic system, in which the desired solution of the corresponding equation is divided up into a series of its simplest solutions; an example of this is the solution of the equation of an oscillating string [Sec. 421]. The growth of mathematical physics continued at the beginning of the nineteenth century and is connected with the names of Poisson, Fourier, Cauchy and Ostrogradski.

The general attempt to increase rigorousness naturally drew attention to the problem of the actual existence of a solution for differential equations (or a system formed from them). For ordinary equations the first results were obtained here by Cauchy, and for equations in partial derivatives by Sophia Kovalevskaya†.

At the end of the last century, in connection with the requirements of mechanics and astronomy, there appear investigations on the *qualitative theory of ordinary differential equations* and *systems of these equations*, which established various properties of the solutions, from the equations themselves, without integrating them. The originators of this theory were Poincaré and Liapounoff‡. Liapounoff's ideas were further developed in the works of Russian mathematicians.

We note further that recently there has been a definite influence on the theory of differential equations (particularly those in partial derivatives) by new analytical disciplines—*the theory of functions of a real variable* and *functional analysis* (see Parts V and VI). This influence is not only reflected in general points of view and in the approach to the fundamental concepts of the theory of differential equations, but has also brought about important concrete results in this theory.

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† Sophia Vasilevna Kovalevskaya (1850–1891) was the first woman to hold a position as a professor of mathematics.

‡ The French academic Henri Poincaré (1854–1912) and the Russian academician Alexander Michailovich Liapounoff (1857–1918) were outstanding mathematicians.

II. Variational calculus

Variational calculus originated at the end of the seventeenth century, soon after "infinitesimal calculus" was created, and together with the latter formed a branch of the so-called *functional analysis*, which has been developed in our own century (see Part VI).

For clarity of presentation, we will first establish a concept which belongs to the realm of functional analysis.

Suppose that we have a plane curve (Fig. 84).

$$y = f(x) \quad (x_0 \leqslant x \leqslant x_1).$$

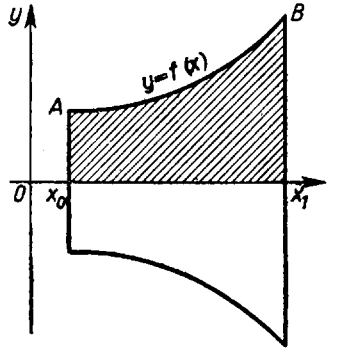

FIG. 84.

We consider: *(1)* the area P of the curvilinear trapezoid, *(2)* the length S of the curve, *(3)* the area Q of the surface, obtained by rotating the curve around the x-axis, and, finally, *(4)* the volume V of the solid of revolution, bounded by this surface. All these are examples of quantities *which are dependent on the form of the curve* or, equivalently, *on the form of the function f, i.e. immediately on all its values.* For the quantities mentioned we had before the integral representations [Secs. 196, 201, 205, 198]:

$$\left.\begin{aligned} P &= \int_{x_0}^{x_1} y(x)\,dx, \qquad S = \int_{x_0}^{x_1} \sqrt{\{1 + [y'(x)]^2\}}\,dx, \\ Q &= 2\pi \int_{x_0}^{x_1} y(x)\sqrt{\{1+[y'(x)]^2\}}\,dx, \qquad V = \pi \int_{x_0}^{x_1} [y(x)]^2\,dx. \end{aligned}\right\} \tag{1}$$

We see that the function y plays the part of the original "argument": *its value determines uniquely the values of each of these quantities.* In such a case, *when the numerical values of a certain quantity are related to the function* $y = y(x)$ *by a definite rule, this quantity is called a functional of the function* y and is denoted by

$$U(y).^{\dagger}$$

All the functionals (1) are contained, as special cases, in the following:

$$U(y) = \int_{x_0}^{x_1} F(x, y, y')dx = \int_{x_0}^{x_1} F'(x, y(x), y'(x))dx, \qquad (2)$$

where by F we understand a given function of three arguments, and $y = y(x)$ is any (continuously differentiable) function of x, for which the expression under the integral and the integral itself have a meaning.

Limiting ourselves for brevity to functionals of the type (2), we can now formulate the simplest variational problem: *from amongst all the smooth curves* $y = y(x)$, *joining the two points* (x_0, y_0) *and* (x_1, y_1), *to find the one for which the functional* (2) *has the greatest (least) value.*

More often, however, a functional is examined at a "local" maximum (minimum)—the value of the functional on the desired curve is compared with its values only on curves sufficiently "near" (in some sense) to the desired curve. It is easy to discern here an analogy with the definition of "local" extremes in the case of a function of one or several variables [Secs. 112, 151].

The beginning of variational calculus is usually dated 1696, when Johann Bernoulli threw down the gauntlet of battle to the mathematicians of his time in the solution *to the problem* (raised by Galileo) *of the brachistochrone* ("the curve of the shortest time"). The essence of the problem is as follows: *of all the lines joining two given points*

† It is impossible to confuse a functional with a function of a function, i.e. with a compound function, which associates with each value of x separately a number, while a functional associates a number with the entire function $y(x)$ as a whole.

A and B (Fig. 85), *not lying in one vertical, it is required to find the curve for which a particle rolls down from the point A to the point B—with no initial velocity and only under the action of the force of gravity—in the shortest time.* The desired curve turned out to be a cycloid, whose base is a horizontal, with a peak at the point *A*.

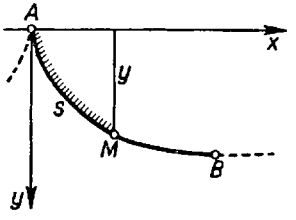

FIG. 85.

We will show that in this problem also we are concerned with a functional of the type (2). Let the coordinate axes be located as they are indicated in the diagram. A particle of mass m varies along an arbitrary curve $y = y(x)$, joining the points $A(0, 0)$ and $B(x_1, y_1)$; from the position A to the position $M(x, y)$, the force of gravity does the work mgy, which—by the law for moving forces—must be equal to the kinetic energy $mv^2/2$ of the point. Hence

$$v = \frac{ds}{dt} = \sqrt{(2gy)}$$

and

$$dt = \frac{ds}{\sqrt{(2gy)}} = \frac{1}{\sqrt{(2g)}} \frac{\sqrt{(1+y^2)}}{\sqrt{y}}\, dx.$$

We integrate with respect to x from 0 to x_1, and then for the time T spent in the whole movement from A to B we finally obtain the expression

$$T = T(y) = \frac{1}{\sqrt{(2g)}} \int_0^{x_1} \frac{\sqrt{(1+y'^2)}}{\sqrt{y}}\, dx \tag{3}$$

of the form (2).

Other problems followed from this; they were all solved differently and each require a special approach: a situation was created which recalled the early history of infinitesimal analysis.

In 1726 in the publications of the St. Petersburg Academy there began a series of works by Euler devoted to extreme problems of this new type. They were summed up in the famous tract *A Method of Finding Curved Lines Possessing Maximal or Minimal Properties, ...* (1744). Here there was given the first "direct" general method for the solution of different types of variational problems. Replacing the curved line by a broken one, Euler reduced the variational problem to the problem of finding the extremes of a function of several variables.

For instance, in the case of the simplest variational problem, we discuss the extreme of the sum

$$\sum_i F\left(x^{(i)}, y^{(i)}, \frac{\Delta y^{(i)}}{\Delta x^{(i)}}\right) \Delta x^{(i)}$$

as a broken function of the ordinates $y^{(i)}$. By equating to zero the derivative of this expression with respect to any ordinate, Euler as a final result arrived at the differential equation of the second order

$$F'_y - \frac{d}{dx} F'_{y'} = 0$$

or, in expanded form,

$$F'_y - F''_{yx'} - F''_{yy} y' - F''_{y'y'} y'' = 0.$$

This equation thus expresses a necessary condition for the curve $y = y(x)$ to provide the extreme of the functional (2). If we succeed in solving Euler's differential equation, taking into account the limit conditions $y(x_0) = y_0$, $y(x_1) = y_1$, then we shall usually obtain at once the desired function. Thus, in the case of the problem of the brachistochrone we can take [see (3)]

$$F = \frac{\sqrt{(1 + y'^2)}}{\sqrt{y}},$$

and Euler's equation assumes the form:

$$2yy'' + y'^2 + 1 = 0.$$

Hence, in lowering the order, it is easy to find the first integral

$$y(y'^2 + 1) = 2C_1,$$

so that

$$dx = \sqrt{\left(\frac{y}{2C_1 - y}\right)}\, dy.$$

Letting $y = C_1(1 - \cos t)$ here, we find that $dx = C_1(1 - \cos t)dt$, hence $x = C_1(t - \sin t) + C_2$. Assuming that when $t = 0$ the curve passes through the origin, we have $C_2 = 0$; the constant C_1 is determined from the requirement that the curve pass through the second of the given points. Thus we have obtained the cycloid mentioned above.

In spite of the fact that Euler's methods were cumbrous and his reasoning imprecise, *these researches laid the foundations of variational calculus as a fundamental mathematical field.*

Somewhat later the then young Lagrange opened a new page in the history of variational calculus by creating a *method of variations*, from which the calculus obtained its name. In 1755 he reported this to Euler, who used the new method in his own later works. The concept of a variation is related to the concept of a differential in ordinary analysis, but it is connected not with the variation of the variable x, but with the variation in the form of the function $y(x)$. If the function $y = y(x)$ provides the functional $U(y)$ with an extreme, then the variation in U, δU, vanishes (again this is only a necessary condition).

Discontinuing at this point the history of variational calculus, we will only remark that Lagrange's theory was extended by Ostrogradski to the case of multiple integrals; subsequently sufficient conditions for an extreme were established; finally, in later decades interest again arose in Euler's ideas, and his methods served as a prototype for the "direct" methods for the solution of variational problems (to a considerable extent in the works of Russian mathematicians).

Almost all the fundamental laws of mechanics and physics are usually formulated as "variational principles", establishing that the variation of some functional or other must be zero. In this connection variational calculus provides the solutions for many important problems of physics and technology.

III. The theory of functions of a complex variable

Complex numbers appeared in mathematical use at the beginning of the sixteenth century, but they became familiar only slowly and with difficulty over the period of a century. Only in the eighteenth century did Euler (again we must mention the name of this brilliant and so widely gifted scholar) begin to consider the complex variable and functions of it. Between his thirtieth and fortieth years he worked out the whole *theory of elementary functions of a complex variable*, using for this purpose not only the apparatus of power series (known since Newton's time), but also infinite products and series of simple fractions. In this way, as we already know [Sec. 254], he established the remarkable connection between exponential and trigonometrical functions.

Another important cycle of works by Euler was related to *integration with respect to a complex variable* (here in the sense of calculating a primitive). In one of these, presented to the St. Petersburg Academy in 1777, but only published in 1793, Euler considered the integral $\int Z(z)\,dz$, taking $Z(z)$ to be a function which is given analytically (for instance, a power series), and he substituted here instead of z the imaginary value $x + yi$†. Then the integral itself takes the form $P(x, y) + Q(x, y)i$, and the expression under the integral is transformed into the following:

$$[M(x, y) + N(x, y)\,i]\,[dx + dy\,i] = [M\,dx - N\,dy] + [N\,dx + M\,dy]\,i.$$

Comparison separately of the real and imaginary parts of the equation obtained leads to the two integrals

$$P = \int M\,dx - N\,dy, \qquad Q = \int N\,dx + M\,dy.$$

Because the differential expressions $M\,dx - N\,dy$ and $N\,dx + M\,dy$ turn out, thus, to be exact differentials (respectively, for P and Q), M and N are necessarily connected by the characteristic relations for exact differentials [Sec. 350]

$$\frac{\partial M}{\partial y} = -\frac{\partial N}{\partial x} \quad \text{and} \quad \frac{\partial N}{\partial y} = \frac{\partial M}{\partial x}. \tag{4}$$

† For brevity, here and from now on we will write i, which is familiar to us, although in its place Euler wrote $\sqrt{-1}$. He soon introduced the notation i, which, however, did not come into general use for some time.

Thus there appeared for the first time in general form relations between the partial derivatives of the real and imaginary parts of a function Z of a complex variable†. These relations were destined to play an important part both in the subsequent development of the actual theory of functions and in its applications.

Cauchy and Riemann shared the fame of the creation of this theory as an independent scientific field, rich in its applications.

At the very beginning of his *Memoranda on Definite Integrals Taken between Imaginary Limits* (1825) Cauchy established the concept of a *definite integral*

$$\int_{x_0+y_0 i}^{X+Y_i} f(z)\,dz \tag{5}$$

as the limit of integral sums—similar to a curvilinear integral [Sec. 330]. However, we find mention of a "curve" elsewhere in the memoranda, and here Cauchy simply considers two monotonic continuous functions having (continuous) derivatives:

$$x = \varphi(t), \quad y = \chi(t) \qquad (t_0 \leqslant t \leqslant T) \tag{6}$$

with the condition that

$$\varphi(t_0) = x_0, \quad \varphi(T) = X; \quad \chi(t_0) = y_0, \quad \chi(T) = Y,$$

and he reduces the integral (5) to an ordinary integral with respect to the real variable t [cf. Sec. 331, (6)]:

$$\int_{t_0}^{T} f(\varphi(t) + \chi(t)i)\,(\varphi'(t) + \chi'(t)i)\,dt.$$

Here, with the hypothesis that the function $f(z)$ has a (continuous) derivative, he establishes the fundamental theorem on *the independence of the integral* (5) *of the choice of the function* (6) (i.e. of the path of integration). Cauchy proved the theorem by the variational method.

With this assumption with respect to the function $f(z)$ Cauchy, in subsequent publications and lectures, established the fact that

† Similar relations, however, occur in the investigations into hydromechanics by d'Alembert (1752) and Euler.

the integral along a closed contour vanishes, and also deduced the famous formula which bears his name:

$$f(z) = \frac{1}{2\pi i} \int_{(C)} \frac{f(t)}{t - z}\, dt,$$

where the integral is taken along the contour (C) of the domain (K) in the complex plane, and z is any interior point of this domain. We encounter here a remarkable property of the class of functions of the complex variable under consideration: *the values of the function inside the domain are uniquely determined by its values on the contour.*

Although Cauchy's formula was proved by him only under the hypothesis that the function $f(z)$ has a (continuous) first derivative in the domain considered, yet from the formula itself, for instance, by means of successive differentiation with respect to z under the integral sign, it can be deduced immediately that *for $f(z)$ its derivatives of all orders exist simultaneously.* Moreover, taking in the domain (K) a circle of radius r with centre at the point z_0, we can assert that *in this circle the function $f(z)$ can be expanded in a series of powers of $z - z_0$, which is its Taylor series* [cf. Sec. 277]. From this follows the more general assertion, made by Cauchy with respect to the radius of convergence of a Taylor series: *this radius is always equal to the distance from the point z_0 to the nearest "singular" point of $f(z)$*—in the neighbourhood of which $f(z)$ represents a single-valued function having a (continuous) derivative.

All this sheds light, for instance, on the enigmatic fact that the function $\arctan x$ and its derivative $1/(1 + x^2)$ can be expanded in a series of powers of x only inside the interval from -1 to $+1$, with radius of convergence 1. On the real axis in general it has no "singularities", but if we go over to the complex domain, then for the corresponding functions in the complex variable z: $\arctan z$ and $1/(1 + z^2)$, there will be "singular" points at the points $z = \pm i$ on the imaginary axis, as a result of which the radius of convergence is equal to the modulus of both these numbers, i.e. 1. Thus the theory of functions of a complex variable enables one sometimes to understand the laws relating to functions of a real variable.

It is useful to give an account of why the requirement that the existence of the derivative of the function $w = f(z)$ with respect to

its *complex* argument z is immeasurably stronger than the formally analogous requirement with respect to a function with a *real argument.* In fact the limit

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{dw}{dz}$$

must be precisely the same, *independent of the direction in which* $z + \Delta z$ *approaches* z. In this lies the true explanation of the remarkable consequences which are implied by the simple fact that there exists a (continuous) derivative of a function with respect to a *complex* argument. The problem of the *conditions for the existence of such a unique derivative* arose comparatively recently: Cauchy approached this problem in 1847, and Riemann began on it in his dissertation *The Elements of the General Theory of Functions of one Complex Variable*†. If we set $z = x + yi$ and $w = u + vi$, where u, v are real functions of two real variables x, y, then the conditions given reduce to the relations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \tag{4a}$$

which the reader will recognize as Euler's equations (4). These relations serve as a basis for the applications of the functions of a complex variable to electrostatics, hydro- and aeromechanics, the theory of elasticity and, finally, the theory of the propagation of heat.

If the complex variables z and w are interpreted as points in the corresponding planes, then the functional dependence between them $w = f(z)$ associates with every point z in the domain $\mathfrak{Z}$ on the plane of the values of the argument a definite point w in the domain $\mathfrak{W}$ on the plane of the values of the function. In other words, the function $w = f(z)$ realizes *a pointwise transformation of the domain* $\mathfrak{Z}$ *into the domain* $\mathfrak{W}$ (Fig. 86). From the existence of a unique derivative for the function $f(z)$ in the domain $\mathfrak{Z}$ (provided it does not vanish) Riemann deduced the following remarkable property of this transformation: *if we take two intersecting curves in the domain* $\mathfrak{Z}$ (zz_1 and zz_2 in Fig. 86), *and the curves corresponding to them in the domain* $\mathfrak{W}$ (ww_1 and ww_2), *then the angle between them will be the*

† See Bernhard Riemann, *Works* (Gostekhizdat, 1948), pp. 49–87.

same in both cases (read in the same direction): the transformation preserves angles or, as we say, is conformal.

Conformal transformations, in connection with the drawing of maps, were studied by Euler, Lagrange and Gauss long before Riemann. But Riemann first clearly connected the property that the function $w = f(z)$ should realize a conformal transformation with the fact that it has a definite derivative. He proved the theorem, which is fundamental in the theory of conformal transformations, *on the possibility of there existing a conformal transformation* (and under definite conditions, a unique one) *of a circle into an arbitrary simply connected domain, and conversely.*

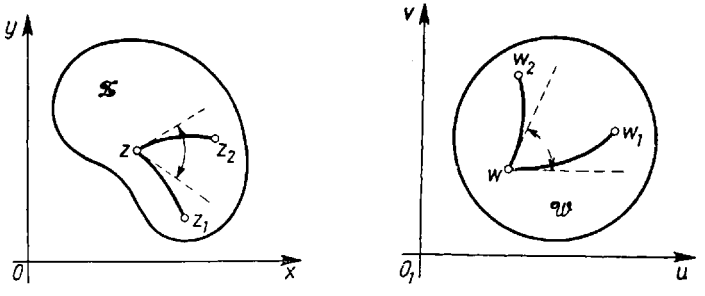

FIG. 86.

Similar transformations are often used in various applications of the theory of functions of a complex variable. This idea is the basis of the theory of the aerofoil originated by Zhukovsky†.

We will not linger over the subsequent brilliant development of the theory of functions of a complex variable.

IV. The theory of integral equations

An equation with respect to a desired function which contains this function under the integral sign is called an *integral equation*‡. In the search for an unknown function it is often more convenient

† Nikolai Yegorovich Zhukovsky (1847–1921)—a famous Russian scientist.

‡ This term was first used in 1888 by du Bua-Ramon; the latter said clearly that progress in the theory of differential equations in partial derivatives is connected with the study of integral equations, "about which, however, nothing is known".

to start from an integral equation rather than from a differential equation. The reader has already encountered such a state of affairs in the proof of the "existence theorem" in the theory of differential equations: there, for instance, the equation $y' = f(x, y)$ together with the initial condition $y(x_0) = y_0$ is replaced by a single integral equation

$$y = \int_{x_0}^{x} f(x, y)\,dx + y_0,$$

to which we can then conveniently apply the method of successive approximations. In our course we have already, formally speaking, dealt with integral equations when we studied Fourier transforms [Sec. 412, formulae (13), (14), (15)].

An important part in attracting the attention of mathematicians to integral equations was played by one mechanical problem, raised and solved by Abel (1826). In this it was required *to find a curve*

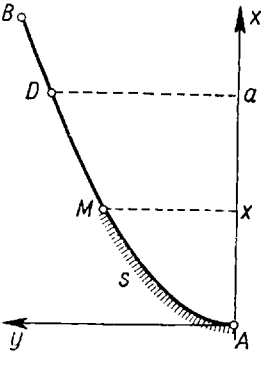

FIG. 87.

AB in the vertical plane (Fig. 87) *such that the time* τ, *in the course of which a heavy material particle, released from any point D without initial velocity, rolls along the curve with its lowest point at A, is a continuous function, given in advance, of the height a of the point D*:

$$\tau = \varphi(a).$$

We arrange the coordinate axes as indicated in the diagram, and we will let the equation of the desired curve be $y = y(x)$. As the point

moves from the original position D to M, by the rule for a moving force, we will have (as above, in the problem of the brachistochrone)

$$\frac{1}{2}mv^2 = mg(a-x), \quad v = \frac{ds}{dt} = -\sqrt{[2g(a-x)]}^{\dagger},$$

whence

$$dt = -\frac{ds}{\sqrt{[2g(a-x)]}}$$

$$\tau = \frac{1}{\sqrt{(2g)}}\int_0^a \frac{\sqrt{\{1+[y'(x)]^2\}}}{\sqrt{(a-x)}}\,dx.$$

Denoting $\sqrt{(1+[y'(x)]^2)}$ by $\sqrt{(2g)}\,u(x)$, to define the unknown function $u(x)$ we will have the integral equation

$$\int_0^a \frac{u(x)\,dx}{\sqrt{(a-x)}} = \varphi(a). \tag{7}$$

Abel in fact solved an even more general equation

$$\int_0^a \frac{u(x)\,dx}{(a-x)^{\alpha}} = \varphi(a) \qquad (0<\alpha<1);$$

for simplicity we will limit ourselves to equation (7).

We multiply both sides of this by $1/\sqrt{(z-a)}$ and we integrate with respect to a from 0 to z:

$$\int_0^z \frac{\varphi(a)\,da}{\sqrt{(z-a)}} = \int_0^z da \int_0^a \frac{u(x)\,dx}{\sqrt{[(z-a)(a-x)]}}.$$

We invert the order of integration of the repeated integral on the right, by formula (9), Sec. 344‡:

$$\int_0^z u(x)\,dx \int_x^z \frac{da}{\sqrt{[(z-a)(a-x)]}}.$$

† We recall that as t increases the arc $s = \cup AM$ decreases.

‡ The domain (P), which was mentioned in Remark I, Sec. 344, is a triangle here, illustrated in Fig. 88. However, since the function under the integral tends to ∞ when $a = x$ or $a = z$, it would be necessary in addition to justify the change of order of the integrals.

Because the inner integral here is equal to π [Sec. 292, *(1)*], we arrive at the equation

$$\int_0^z \frac{\varphi(a)\,da}{\sqrt{z-a}} = \pi \int_0^z u(x)\,dx$$

and, differentiating with respect to z, we finally obtain the following expression for the desired function:

$$u(z) = \frac{1}{\pi}\frac{d}{dz}\int_0^z \frac{\varphi(a)\,da}{\sqrt{z-a}}.$$

Knowing $u(x)$, it is then not difficult to determine $y(x)$.

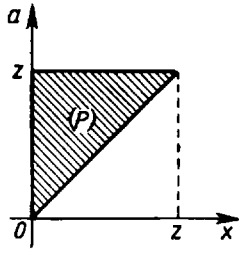

FIG. 88.

Abel's equation—retaining the principle of his method of solution—was generalized considerably later (1884) by Sonin†. The problem was completed when in 1896 Volterra‡ began to publish a series of researches in which he presented the general theory of integral equations of the form

$$\int_a^x N(x,s)\varphi(s)\,ds = F(x) \quad \text{and} \quad \varphi(x) + \int_a^x N(x,s)\varphi(s)\,ds = F(x),$$

which were afterwards called "Volterra integral equations", respectively, of the first and second type. Here the required function is $\varphi(x)$, and $N(x,s)$ and $F(x)$ are the given functions. Soon after this,

† The academician Nikolai Yakovlevich Sonin (1849–1915).

‡ Vito Volterra (1860–1940)—an Italian mathematician.

in the period from 1900 to 1903, there appeared some remarkable works by Fredholm† devoted to the equations

$$\int_a^b N(x, s)\varphi(s)ds = F(x) \quad \text{and} \quad \varphi(x) + \int_a^b N(x, s)\varphi(s)ds = F(x).$$

These "Fredholm equations" of the first and second types differ from the "Volterra equations" in that here both limits in the integrals are constants. (However, we can regard the Volterra equations as a special case of the Fredholm equations, if we set $N(x, s) = 0$ for $s > x$.)

Without embarking on an exposition of Fredholm's theory, we will limit ourselves to saying that it represents, in a certain sense, a limit case of the theory of algebraic linear equations.

During the course of the first half of this century the theory of integral equations has obtained wide applications in various directions and has acquired importance as a powerful means for investigating the phenomena of nature.

V. The theory of functions of a real variable

Although our whole course in mathematical analysis has been devoted to the study of functions of real variables, yet by the name indicated in the title we understand not this "classical" part of analysis, but a more profound and subtle theory, which appeared only at the beginning of this century. However, its origins can be discerned in the critical direction in mathematics, arising at the end of the eighteenth and the beginning of the nineteenth centuries, which we have mentioned more than once. *The raising of the requirements of rigorousness in mathematical proofs compelled mathematicians to doubt a number of naive arguments of earlier mathematicians.* For instance, the attempts of Ampère (1806) and several other authors to prove the differentiability of any continuous function, excluding perhaps only separate points, provoked the examples of Bolzano, Weierstrass and others [cf. Sec. 271] contradicting this. On the other hand, *the increasing tendency to generalization created an*

† Erik Ivar Fredholm (1866-1927)—a Swedish mathematician.

interest in functions with different "pathological" peculiarities, often running contrary to our intuitions.

All this tendency, which is of a negative character (and which, by the way, aroused bitter opposition on the part of the representatives of "classical" analysis), belongs to the foundations of scientific discipline as we study it now. The transition from criticism of the old and pathological enthusiasm to a positive theory was to a large degree stimulated by *the theory of infinite sets* created in the seventies and eighties of the last century by Georg Cantor. His ideas, which encountered opposition at first on the part of his contemporaries, afterwards had a great influence on the development of many domains of mathematics, in particular on *the theory of functions of a real variable.*

This last was formulated as an independent scientific discipline at the turn of the nineteenth and twentieth centuries in the works of the French mathematical schools, representatives of which were Emile Borel (1871–1956), René Baire (1874–1932) and Henri Lebesgue (1875–1941).

An important part was played by the concept introduced by Borel (1898) and Lebesgue (1902) of *the measure of a linear point set,* applicable to a very wide class of so-called *measurable* sets. The measure mE of a point set E generalizes the concept of the length of an interval. Similar to a length, it possesses the additive property, but to a still wider degree: in fact, if a bounded set E is the join of a finite number or even an infinite (but in this case necessarily a countable) set of measurable sets

$$E_1, E_2, \ldots, E_n, \ldots,$$

having no common points in pairs, then E is measurable, where the *measure of E is equal to the sum* (or the sum of the series) *of the measures of the sets E_n*:

$$mE = \sum_n mE_n.$$

If the point set E can be covered by a finite or a (countably) infinite system of intervals with arbitrarily small sum of their lengths, then its measure is equal to zero. Let the function $f(x)$, given in the

interval $[a, b]$, possess a definite property at all points of the interval with the exclusion of, at most, the points of some set of zero measure; then we say that the function has this property "almost everywhere" in the given interval. We use this expression in formulating the following two theorems of Lebesgue:

I. *The function $f(x)$, monotonic in the interval $[a, b]$, has two-sided finite derivatives almost everywhere in this interval.*

II. *A continuous rectifiable curve $y = f(x)$ $(a \leqslant x \leqslant b)$ has a definite tangent almost everywhere.*

These theorems are mentioned as examples of the positive results of the new theory; as we see, there is no mention in them of "pathology", and they establish general and very subtle laws, and indeed in realms where the methods of classical analysis are useless.

The generalized concept of a definite integral, due to Lebesgue, has been of great value in contemporary analysis. The reader should recall Riemann's definition of this concept [Sec. 176]. For the case of a continuous function† it is completely natural: speaking in geometrical terms, for a small Δx_i all the ordinates of the curve $y = f(x)$, corresponding to the change in the abscissa between x_i and x_{i+1}, differ little from an arbitrary one of them, $f(\xi_i)$, and, by replacing approximately the elementary strip by an elementary rectangle with area $f(\xi_i) \cdot \Delta x_i$, we can expect that the limit of the sum

$$\lim_{\Delta x_i \to 0} \sum_i f(\xi_i) \cdot \Delta x_i$$

gives precisely the value of the area under the curve. If, however, the function $f(x)$ is discontinuous, then there is no justification for the given replacement. In reality Riemann's definition is formally applicable only to a narrow class of discontinuous functions, in a certain sense "differing little" from continuous ones (and in fact, as Lebesgue showed, only to functions which are continuous "almost everywhere").

In this connection it is appropriate to recall that if the function $F(x)$ has a continuous derivative $F'(x)$ in the interval $[a, b]$, then

† We recall that in fact for this case Cauchy stated it long before Riemann.

from this derivative the primitive function can easily be "restored" with the help of the definite integral [Secs. 155 and 183, (*12*)]:

$$F(x) = (R)\int_a^x F'(x)\,dx + C. \tag{8}$$

However, in the case of a discontinuous (even if bounded) derivative this formula may be inapplicable simply because the Riemann integral for $F'(x)$ does not exist. Volterra constructed such an example in 1881.

All these circumstances served as a basis for the creation of a new, generalized, concept of a definite integral.

We will explain the fundamental idea of Lebesgue's definite integral for a bounded function $f(x)$:

$$A < f(x) < B \quad (a \leqslant x \leqslant b).$$

Lebesgue divided up into sections, not the interval of variation of the unknown variable, but the interval $[A, B]$, in which are contained the values of the function itself:

$$A = y_0 < y_1 < y_2 < \ldots\ y_i < y_{i+1} < \ldots < y_n = B,$$

and he considered (when $i = 0, 1, \ldots, n-1$) the set E_i of those points at which the values of the function lie between y_i and y_{i+1}:

$$y_i \leqslant f(x) < y_{i+1}.$$

For a small $\Delta y_i = y_{i+1} - y_i$ all these values, in fact, are near each other and differ only a little, for instance, from y_i. Replacing them all by y_i and multiplying this number by the measure of the set E_i, we form the sum

$$\sum_i y_i m E_i. \tag{9}$$

Of course, here we must presuppose that the set of points x from $[a, b]$ for which the values of the function $f(x)$ lie between some numbers α and β $(\beta > \alpha)$ is always measurable; Lebesgue only considered such functions, calling them *measurable*. For the sum (9), whatever may be the bounded measurable functions $f(x)$, there always exists a finite limit as $\Delta y_i \to 0$; it represents *the Lebesgue integral of the function $f(x)$ from a to b* and is denoted by the symbol

$$(L)\int_a^b f(x)\,dx.$$

The domain of applicability of the Lebesgue integral is incomparably wider than that of the Riemann integral. In particular, with its help we can always "restore" the primitive function $F(x)$ from its bounded derivative $F'(x)$:

$$F(x) = (\text{L})\int_a^x F'(x)\,dx + C. \tag{8a}$$

Lebesgue's definition can also be extended to the case of an unbounded function; the integral obtained here is absolutely convergent. Afterwards there were proposed further generalizations of the concept of an integral, for which the integral may converge non-absolutely, but none of them found such wide and fruitful applications in the various domains of analysis as the Lebesgue integral.

Further progress in the theory of functions of a real variable is essentially connected with the work of the Moscow Mathematical School, the founders of which were Dmitri Fedorovich Yegorov (1869–1931) and the academician Nikolai Nikolaevich Luzin (1883–1950). We quote below, in the form of examples, two fundamental theorems: the first is due to Yegorov, and the second to Luzin. They show that—in spite of the extreme generality of the objects studied in the contemporary theory of functions—the connection with classical analysis, in a sense, has not been lost. The theorems are:

I. *The convergence of any sequence* $\{f_n(x)\}$ *of measurable functions to a limit function* $f(x)$ *is uniform if we disregard a certain point set of arbitrarily small measure.*

II. *Any measurable function* $f(x)$ *is continuous if we disregard a certain point set of arbitrarily small measure.*

The merit of the Moscow school is not only the further development of the actual theory of functions of a real variable, but also the instilling of its concepts and methods in other domains of analysis, and also in the theory of probability.

Until now we have only discussed the so-called *metric theory of functions*, using essentially the concept of the measure of a point set. Another branch of the new theory of functions, called *descriptive*, originated in the work of Baire (1899) and Lebesgue (1905), who

studied the structure and properties of discontinuous functions from the point of view of their formation from continuous functions by means of successive transitions to limits. These investigations are closely connected with *the descriptive theory of sets*, where more and more complicated classes of point sets are always being studied; these are obtained by means of a series of very simple operations from intervals (rectangles and so on). An important part in the further extension and amplification of this theory is also due to Luzin and his investigations.

We will pause over one further branch of mathematical analysis, called *the construction theory of functions*. It is concerned with the problems of the approximate representation of a function by means of various kinds of analytical apparatus, the simplest examples of which are algebraic or trigonometrical polynomials.

Pafnutii Lvovitch Tchebychev in his work *The Theory of Mechanisms, known as Parallelograms*† (1854) noted that the segment of the Taylor series of the function $f(x)$

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

represents the function better than other polynomials of the same degree *only near the point* $x = x_0$. If, however, we are discussing the approximate representation of a function by a polynomial of the nth degree in a *fixed interval* $[a, b]$, then we must prefer another polynomial $P_n(x)$ to the Taylor polynomial, provided that "the limit of its deviation from $f(x)$ in the given interval is less than the limits of the deviations of all other polynomials of the same degree". Thus, the problem was first raised of the "polynomial of the best approximation", determined from the condition that

$$\max_{a \leqslant x \leqslant b} |P_n(x) - f(x)|$$

has the least possible value; the latter is usually denoted by $E_n(f)$. Tchebychev's investigations served as a starting point for a vast series works by the St. Petersburg school.

† The complete collected works of P. L. Tchebychev, Vol. II (publ. Acad. Nauk, 1947), pp. 23–51.

In 1885 Weierstrass published a theorem [Sec. 278], from which it appeared that for any function $f(x)$ continuous in $[a, b]$

$$\lim_{n\to\infty} E_n(f) = 0;$$

of course, it follows from the existence of the last relation that the function f is continuous [Sec. 266]. This was a simple example of the mutual connection between the structure property of the function f (continuity) and the behaviour of the quantity $E_n(f)$ (its tendency to zero).

The ideas of Tchebychev and Weierstrass brought about an essential development in the researches of the academician Sergei Natanovich Bernstein (born in 1880) and other Russian mathematicians, and also of the American scholar D. Jackson (1888–1946). The idea of how the differential and other properties of the function $f(x)$ influence the speed with which the quantity $E_n(f)$ tends to zero, and conversely, runs through these works.

VI. Functional analysis

This is the youngest of all branches of mathematical analysis: it is only a very few decades old. The origins of functional analysis were promoted by constant remarkable analogy between objects which seem remote at first sight, which incited mathematicians to try to find both the concepts joining them, and general arguments which would cover the special arguments for these objects, using only the relation which in fact exists between them.

It is remarkable here that the manifolds of objects considered usually resemble, in some respect or other, real space. These manifolds are also called *spaces*, although their elements may be—according to the case considered—objects of a uniform nature: curves or spaces, functions of one or several continuously varying arguments, finite systems or sequences of real numbers (i.e. functions of a natural argument), and so on. We have already encountered similar "spaces" in Sec. 125 (arithmetical m-dimensional space) in Sec. 418 (the space of continuous functions) and, finally, in part II, devoted to variational calculus (the space of smooth curves or continuously differentiable functions).

In such a space $X = \{x\}$ we will consider and investigate the *functional operator*

$$y = U(x) \tag{10}$$

(a generalization of our ordinary concept of a "function"), which relates, by some rule or other, to every element x of X a definite element y of some space or other, Y. In the special case when Y is simply a set of real numbers, so that the element x is related to a number, the functional operator is called a *functional*—we have already discussed this concept above (see part II).

We will stop to consider only one important type of space, namely *the metric spaces*. This is the name we give to the space X if, for every pair of elements x and $\bar{x}$ belonging to it, there is defined a *distance* (a "metric function"), i.e. a non-negative real number $\varrho(x, \bar{x})$, having the following properties:

(1) $\varrho(x, \bar{x}) = 0$ is equivalent to $x = \bar{x}$;

(2) $\varrho(\bar{x}, x) = \varrho(x, \bar{x})$,

and, finally,

(3) $\varrho(x, \bar{\bar{x}}) \leqslant \varrho(x, \bar{x}) + \varrho(\bar{x}, \bar{\bar{x}})$ [the "triangle axiom"; cf. Secs. 125 and 418].

The convergence of the sequence $\{x_n\}$ to the limit element x is defined by the relation:

$$\varrho(x_n, x) \to 0;$$

here we write

$$x = \lim x_n \quad \text{or} \quad x_n \to x.$$

We will give some examples of metric spaces.

A. Arithmetical space—linear or, in general, m-dimensional. In the linear case the elements x are real numbers, and

$$\varrho(x, \bar{x}) = |x - \bar{x}|.$$

When $m > 1$ the elements x are ordered systems of real numbers: $x = (\xi_1, \xi_2, \ldots, \xi_m)$, where the ξ_i are the "coordinates" of the point x. In this case

$$\varrho(x, \bar{x}) = \sqrt{[(\xi_1 - \bar{\xi}_1)^2 + \ldots + (\xi_m - \bar{\xi}_m)^2]},$$

if by $\bar{\xi}_i$ we denote the "coordinates" of the point $\bar{x}$.

B. The space of functions $x = x(t)$ continuous in the interval $[a, b]$ with the metric

$$\varrho(x, \bar{x}) = \max_{a \leqslant t \leqslant b} |x(t) - \bar{x}(t)|.$$

C. The same manifold, but with another metric:

$$\varrho(x, \bar{x}) = \sqrt{\left\{\int_a^b [x(t) - \bar{x}(t)]^2 \, dt\right\}}$$

[cf. Sec. 418, Remark].

D. The space of the sequences of real numbers

$$x = (\xi_1, \xi_2, \ldots, \xi_n, \ldots),$$

for which the series $\sum_1^\infty \xi_n^2$ converges. The metric:

$$\varrho(x, \bar{x}) = \sqrt{\left[\sum_1^\infty (\xi_n - \bar{\xi}_n)^2\right]}.$$

E. The space of bounded sequences of real numbers with the metric

$$\varrho(x, \bar{x}) = \sup_n |\xi_n - \bar{\xi}_n|.$$

It is easy to show, by using the "triangle axiom", that, for any metric space, from the convergence of a sequence of elements $\{x_n\}$ to a limit element x follows the fulfilment of a condition analogous to the well-known Bolzano–Cauchy condition [cf. Sec. 52]:

$$\varrho(x_n, x_{n'}) < \varepsilon \quad \text{provided only} \quad n, n' > N_\varepsilon.$$

The converse conclusion—that there exists a limit element x for every sequence $\{x_n\}$ satisfying this condition—is not true for all metric spaces. Those spaces for which it is true are called *complete*; such, for instance, are the spaces A, B, D, E; the space C is not complete†.

Now, in order that the reader should understand clearly the usefulness of this great degree of abstraction and generalization which exists in functional analysis, we will give by way of an example one

† It can be made complete, if we add to it all the discontinuous functions $x(t)$, integrable in Lebesgue's sense, together with their squares. Here two functions which coincide "almost everywhere" are regarded as one element of the space.

theorem of the well-known Polish mathematician Stefan Banach (1892–1945), one of the originators of contemporary functional analysis. It states:

Let there be given a functional operator (10), *defined in a complete metric space* $X = \{x\}$ *and mapping an element* x *of* X *again into an element of the same space. If here it is always true that*

$$\varrho(U(x),\ U(\overline{x})) \leqslant \alpha\varrho(x, \overline{x}) \quad \text{where} \quad 0 < \alpha < 1,$$

there exists a (unique) point x_0 *such that*

$$U(x_0) = x_0. \tag{11}$$

This point is called the fixed point of the operator U; it is none other than the "root" of the *functional equation*

$$x = U(x).$$

The following consideration serves as a guide to the proof. If x, any point but x_0, is a fixed point (we presuppose that such a point exists), then

$$\varrho(U(x),\ U(x_0)) = \varrho(U(x),\ x_0) \leqslant \alpha\varrho(x,\ x_0),$$

so that the element $x_1 = U(x)$ is nearer to x_0 than x. It is natural to repeat this process, starting from x_1:

$$\varrho(U(x_1), x_0) \leqslant \alpha\varrho(x_1, x_0) \leqslant \alpha^2\varrho(x, x_0),$$

and the element $x_2 = U(x_1)$ is even nearer to x_0. This also suggests the idea of starting from an arbitrarily chosen element x and constructing, by repeatedly applying the operator U, a sequence of elements

$$x_1 = U(x),\ x_2 = U(x_1),\ \ldots,\ x_n = U(x_{n-1}),\ \ldots. \tag{12}$$

For this sequence, $\{x_n\}$, it can be proved that the Bolzano–Cauchy condition is satisfied, whence (in view of the completeness of the space) it follows that it converges to some limit element x_0. Then it can be established that this is the required fixed element (and moreover is unique). This is the plan of the proof, which we will not give in detail.

We will only draw the reader's attention to the application, for the x_0 obtained, of the method of successive approximation by the scheme (12). It is easy to obtain an estimate to the nth approximation:

$$\varrho(x_n,\ x_0) \leqslant M\alpha^n \quad (M = \text{const}).$$

Now we will enumerate several problems from different fields which can be solved at once by the use of the "principle of the fixed point" which we have established.

(1) "Kepler's equation" well known in theoretical astronomy

$$x = m_0 + \varepsilon \sin x \tag{13}$$

defines the so-called eccentric anomaly x of a planet given its mean anomaly m_0 and the eccentric of the planet orbit ε $(\varepsilon < 1)$.

Let the "space" X be simply the set of all real numbers (see A), and let the "functional operator" be the ordinary function of the numerical variable x:

$$U(x) = m_0 + \varepsilon \sin x.$$

It is clear that the fixed point x_0 of this "operator" is precisely the desired root of equation (13).

Because

$$\varrho(U(x), U(\bar{x})) = |U(x) - U(\bar{x})| = \varepsilon |\sin x - \sin \bar{x}|$$
$$\leqslant \varepsilon |x - \bar{x}| = \varepsilon \varrho(x, \bar{x}),$$

the condition of the theorem is satisfied. Thus, we arrive at the assertion of the existence of a unique root x_0 of equation (13), where its evaluation can be completed by the method of successive approximation.

(2) Let there be given a Fredholm integral equation of the second type, containing an arbitrary numerical parameter λ:

$$x(s) = f(s) + \lambda \int_a^b K(s, t) x(t) dt, \tag{14}$$

where $f(s)$ is continuous in the interval $[a, b]$, and $K(s, t)$ is continuous in the square $[a, b; a, b]$, and let us suppose that we are looking for a continuous solution $x(s)$ of this equation. Here the space X is the space of continuous functions in $[a, b]$ with the metric

$$\varrho(x, \bar{x}) = \max_{a \leqslant t \leqslant b} |x(t) - \bar{x}(t)|$$

(see B). The operator

$$U(x) = f(s) + \lambda \int_a^b K(s, t) x(t) dt$$

relates every continuous function $x(t)$ to some continuous function of s. Here the fixed point $x_0(t)$ of this operator will clearly be the solution of equation (14).

We will verify that the condition of the theorem is satisfied. By the boundedness of the continuous function K:

$$|K(s, t)| \leqslant M \quad (M = \text{const})$$

we will have:

$$\varrho\big(U(x), U(\bar{x})\big) = |\lambda| \max_{a \leqslant s \leqslant b} \left| \int_a^b K(s, t)|x(t) - \bar{x}(t)|\, dt \right|$$

$$\leqslant |\lambda| M(b-a) \max_{a \leqslant t \leqslant b} |x(t) - \bar{x}(t)| = |\lambda| M(b-a) \cdot \varrho(x, \bar{x}).$$

Hence it is clear that, provided only that

$$|\lambda| < \frac{1}{M(b-a)},$$

the condition of the theorem is satisfied. Hence, for sufficiently small (in absolute magnitude) values of the parameter λ equation (14) has a unique solution $x_0(t)$, which can be obtained by the method of successive approximation.

(3) As a last example we will consider the problem of the solution of a system of an infinite (countable) set of linear equations with an infinite (again countable) set of unknowns; such a system can be written in the form:

$$\xi_i = \sum_{k=i}^{\infty} a_{i,k} \xi_k + b_i \quad (i = 1, 2, 3, \ldots) \tag{15}$$

where $a_{i,k}$ and b_i are given coefficients, and the ξ_i are the unknowns. We presuppose that the coefficients $a_{i,k}$ and b_i satisfy the conditions

$$\sum_{k=1}^{\infty} |a_{i,k}| \leqslant \alpha < 1, \quad |b_i| \leqslant \beta \quad (i = 1, 2, 3 \ldots). \tag{16}$$

We will be interested in a bounded solution of the system (15), i.e. a bounded sequence of numbers $\{\xi_i\}$ which satisfies all the equations (15).

It will probably be clear to the reader that the basis of this is to take the space X of bounded sequences $x = \{\xi_i\}$ with the metric

$$\varrho(x, \bar{x}) = \sup_i |\xi_i - \bar{\xi}_i|$$

(see E). The equation

$$\eta_i = \sum_{k=1}^{\infty} a_{i,k}\xi_i + b_i \qquad (i = 1, 2, 3, \ldots) \tag{17}$$

relates each bounded set $x = \{\xi_k\}$ to, by virtue of (16), a bounded set $y = \{\eta_i\}$, i.e. an element of the space X; in this way an operator $U(x)$ is defined. If we take the element $\bar{x} = \{\bar{\xi}_k\}$, then corresponding to this value of the operator $U(\bar{x})$ will be the element $\bar{y} = \{\bar{\eta}_i\}$, defined by equations analogous to (17). Because [by the first of equations (16)]

$$\varrho(y, \bar{y}) = \sup_i \left\{ \left| \sum_{k=i}^{\infty} a_{i,k}(\xi_k - \bar{\xi}_k) \right| \right\} \leqslant \alpha \sup_k |\xi_k - \bar{\xi}_k| = \alpha\varrho(x, \bar{x}),$$

then—all by the same theorem—the desired solution exists and is unique. To obtain this we can use once again the method of successive approximation.

It is very instructive to consider the possibility of solving such different problems, starting from one source. This example will enable the reader to understand more easily how the general theorems of functional analysis can be applied in very different fields. The force of this method, which seems so abstract, lies in the fact that it deals with objects and relations between them in the simplest form, abstracted from their special peculiarities which are inessential for the discussion. How often in the past progress in mathematics has come about in such a way!

In order to emphasize the value of functional analysis on the whole (we have in fact mentioned only its elements), it is sufficient to mention that functional analysis, in its more advanced forms, is used essentially in contemporary *theoretical physics*, and fundamentally in *quantum mechanics*, which, in turn, undoubtedly influenced the development of functional analysis.

Thus, by repeatedly overcoming the mistakes and fallacies of separate investigators and even of whole generations, mathematical analysis up toour time has continued to grow and to develop, with ever more new and living branches. Other new branches of analysis arise immediately from the requirements of the knowledge of the world surrounding us; in other cases they appear, subject to the natural law of development of analysis itself, and only later are enriched by applications. Different branches of analysis gradually penetrate one another and, while keeping the speciality of their own problems and methods, they all form one whole.

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